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ON STABILITY OF SOME TYPES OF FUNCTIONAL EQUATIONS

MASTER THESIS

PRESENTED BY
REHAB SALEEM AL-MOSADDER

SUPERVISED BY
D.r AS'AD Y. AS'AD

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Abstract

In this thesis, we study a stability of some types of functional equations. Functional equations are equations in which the unknown (or unknowns) are functions. the aim of this study investigate Hyers-Ulam-Rassias stability of the orthogonally Jensen functional equation in two kinds (additive and quadratic). And study a special case of the Hyers-Ulam stability problem, which is called the superstability. In this study we investigate the superstability of the pexiderized cosine functional equation

$$f_1(x+y) + f_2(x-y) = 2g_1(x)g_2(y),$$

where f_1, f_2, g_1 and g_2 are functions from $\mathbb{R}$ to $\mathbb{C}$. And we get a some critical values of the stability problem, which is a values that is no stability at it.

We study the two papers titled " The Stability of the Pexiderized Cosine Functional Equation " by C. Kusollerschariya and P. Nakmahachalasint. [3] and " On the Stability of Orthogonal Functional Equations " by Choonkil Park. [1].

Introduction

The subject of functional equations forms a modern branch of mathematics. Functional equations are equations in which the unknowns are functions. and to solve a functional equation means to find all functions that satisfy the functional equation. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be an additive function if it satisfies the additive Cauchy functional equation $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. The simplest and most elegant variation of the additive Cauchy equation is Jensens functional equation. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be Jensen if it satisfies

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}, \quad \forall\, x, y \in \mathbb{R}.$$

Also above equation is called Jensen additive functional equation, and the Jensen quadratic functional equation is given by the following equation

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y)$$

for all $x, y \in X$. In 1903, J.V. Pexider considered another kinds of equations, that equations with several unknown functions in one variable, they called pexider's (Pexiderization) equations. An intriguing and famous talk presented by Stanislaw M. Ulam in 1940, triggered the study of stability problems for various functional equations. Ulam discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms.

Let G_1 be a group and let G_2 be a metric group with a metric $d(.,.)$ Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? If the answer is affirmative, we say that the functional equation for homomorphisms is stable.

In 1941, D. H. Hyers was the first mathematician to present the result concerning the stability of functional equations. He brilliantly answered the question of Ulam for the case where G_1 and G_2 are assumed to be Banach spaces. This result of Hyers is stated as follows

Let $f : E \to E'$ be a function from a Banach space to a Banach space which satisfies the inequality $\|f(x+y) - f(x) - f(y)\| \leq \epsilon$ for all $x, y \in E$. Then there exists a unique additive function ϕ satisfying the inequality $\|f(x) - \phi(x)\| \leq \epsilon$. In 1978, a generalized version of Hyers' result was proven by Th. M. Rassias in [2] where $f : E \to E'$ satisfies the inequality $\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in E$ and for some constants $\theta \geq 0$ and $0 \leq p < 1$. In 1979, J. Baker, J. Lawrence, and F. Zorzitto in [8] introduced that if f satisfies the inequality $|E_1(f) - E_2(f)| \leq \epsilon$, then either f is bounded or $E_1(f) = E_2(f)$, where $E_1(f)$ is the left said of the given functional equation and $E_2(f)$ is the right said of them. This concept is now known as the superstability. In 1980, J. A. Baker in [8] observed the superstability of the well-known cosine functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y) \qquad \forall x,\ y \in \mathbb{R}.$$

In 2008, C. Kusollerschariya and P. Nakmahachalasint [1] investigated the superstability of the following pexiderized cosine functional equation

$$f_1(x+y) + f_2(x-y) = 2g_1(x)g_2(y),$$

where f_1, f_2, g_1 and g_2 are functions from $\mathbb{R}$ to $\mathbb{C}$.

R. Ger and J. Sikorska [11] investigated the orthogonal stability of the Cauchy functional equation, they showed that if f is a function from an orthogonality space X in to real Banach space Y and $\|f(x+y) - f(x) - f(y)\| \leq \epsilon$ for all $x, y \in X$ with $x \perp y$ and some $\epsilon > 0$, then there exists exactly one orthogonally additive mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{16}{3}\epsilon$ for all $x \in X$.

The first author treating the stability of the quadratic functional equations was F. Skof [5] by proving that if f is a mapping from a normed space X into a Banach space Y satisfying $\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \epsilon$ for some $\epsilon > 0$, then there is a unique quadratic function $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{\epsilon}{2}$.

In 2008, Choonkil Park and Themistocles M. Rassias [1] proved the generalized Hyers-Ulam stability of the orthogonally Jensen additive functional equation and of the orthogonally Jensen quadratic functional equation.

This thesis is about stability of some types of functional equations, where we introduce this thesis in three chapters.

Chapter one is titled by some types of functional equations, where we includes preliminaries and definitions that will be used in the remainder of the thesis. Section one contains basic definitions and concepts for abstract algebra that we used. In the second section, we talk about essentialy concepts and solutions of the basic functional equations. In the third section, we give some examples of functional equations. We include this section four different types of functional equations, namely Cauchy, Jensen, Quadratic and Cosine functional equation, we study a particular solutions,

general solutions, the stability problem.
Chapter two is titled by on the stability of orthogonal functional equations, where we contains two sections. In section one, we prove the generalized Hyers-Ulam stability of the orthogonally Jensen additive functional equation. In section two we prove the generalized Hyers-Ulam stability of the orthogonally Jensen quadratic functional equation.
Chapter three, we focus the study on the pexiderized cosine functional equation. In section one, we give definition, example about the superstability concept. In section two, we will prove many of the theorems being studied the superstability of the pexiderized cosine functional equation.

Chapter 1

Some Types of Functional Equations

1.1 preliminaries

Definition 1.1.1. [4] A **metric space** is a pair (X, d), where X is a set and d is a metric on X (or distance function on X), that is; a real valued function defined on $X \times X$ such that for all $x,\ y,\ z \in X$ we have:

(1) $d(x, y) \geq 0$.

(2) $d(x, y) = 0$ if and only if $x = y$.

(3) $d(x, y) = d(y, x)$. (Symmetry)

(4) $d(x, y) \leq d(x, z) + d(z, y)$. (Triangle inequality)

Definition 1.1.2. [4] The space X is said to be complete if every Cauchy sequence in X is converges, that is; has a limit which is an element of X.

Definition 1.1.3. [4] A **normed space** X is a vector space with a norm defined on it. A Banach space is a complete normed space. Here a norm on a (real or complex) vector space X is a real-valued function on X whose value at an $x \in X$ is denoted by $\|x\|$, (read "norm of x") and which has the properties:

(1) $\|\mathbf{x}\| \geq 0$. *(positivity)*

(2) $\|\mathbf{x}\| = 0 \Leftrightarrow x = 0$.

(3) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. *(Triangle inequality)*

(4) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$. *(positive scalability)*

where x and y are arbitrary vectors in X and α is any scaler. A norm on X defines a metric d on X which is given by

$$d(x, y) = \|x - y\| \qquad \forall x, y \in X,$$

and is called the metric induced by the norm. The normed space just defined is denoted by $(X, \|.\|)$ or simply X.

Definition 1.1.4. [9] Let G be a nonempty set together with a binary operation (usually called multiplication) that assigns to each ordered pair (a,b) of elements of G an element in G denoted by ab. We say G is a **group** under this operation if the following three properties are satisfied:

(1) (associative law) for all $a, b, c \in G$,

$$(ab)c = a(bc).$$

(2) (existence of an identity element) there exists an element $e \in G$ (called the identity) in G such that

$$ae = a = ea$$

for all $a \in G$.

(3) (existence of inverses) for each $a \in G$, there exists an $b \in G$ (called an inverse of a) such that

$$ab = e = ba.$$

Definition 1.1.5. [9] A **homomorphism** ϕ from a group G to a group $\bar{G}$ is a mapping from G in to $\bar{G}$ that preserves the group operation, that is;

$$\phi(ab) = \phi(a)\phi(b)$$

for all a, b in G.

Definition 1.1.6. [15] A **vector space** over a field F is a set V together with two binary operations that satisfy the following condition :

(1) $u + (v + w) = (u + v) + w$. (Associativity of addition)

(2) $u + v = v + u$. (Commutativity of addition)

(3) There exists an element $0 \in V$, called the zero vector, such that $v + 0 = v$ for all $v \in V$. (Identity element of addition)

(4) For every $v \in V$, there exists an element $-v \in V$, called the additive inverse of v, such that $v + (-v) = 0$. (Inverse elements of addition)

(5) $a(u + v) = au + av$. (Distributivity of scalar multiplication with respect to vector addition)

(6) $(a + b)v = av + bv$. (Distributivity of scalar multiplication with respect to field addition)

(7) $a(bv) = (ab)v$. (Compatibility of scalar multiplication with field multiplication)

(8) $1v = v$,. where denotes the multiplicative identity in F (Identity element of scalar multiplication)

Where u, v and w be arbitrary vectors in V and a, b are scalars in F.

1.2 Some Examples of functional Equations

In this section we introduce some basic definitions, solutions and stability problems of functional equations. Also we refer to some critical values for this functional equations.

Definition 1.2.1. [10] **Functional equations** are equations in which the unknown or (unknowns) are functions.

There are three subjects in the study of functional equations:

(1) finding regular (particular) solutions.

(2) finding general solutions.

(3) stability problems.

Definition 1.2.2. [2] We say that a function or a set of functions is a **particular solution** of a functional equation or system if, and only if, it satisfies the functional equation or system in its domain of definition.

Definition 1.2.3. [2] Given a class of functions **F**, the **general solution** of a functional equation or system is the totality of particular solutions in that class.

The stability problem [12]

Very often instead of a functional equation, we consider a functional inequality and one can ask the following question: when can one assert that the solutions of the inequality lie near to the solutions of the equation? (In the next sections we discuss in details some stability problems).

For example one of the stability problem had been formulated by S. M. Ulam, in(1940). Let G_1 be a group and let G_2 be a metric group with the metric d. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies the inequality

$$d[f(xy), f(x)f(y)] < \delta \text{ for all } x, y \in G_1,$$

then there exists a homomorphism $H : G_1 \to G_2$ with

$$d[f(x), H(x)] < \epsilon \qquad \text{for all } x \in G_1?$$

These kinds of questions form the material of the stability theory. For Banach spaces, the above problem was solved by D.H. Hyers (1941) with $\delta = \epsilon$ and

$$H(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}.$$

1.2.1 Cauchy's Equations

A. L. Cauchy [12] introduced The following functional equations, they referred to as Cauchy's equations:

$$f(x+y) = f(x) + f(y), \ (additive) \tag{1.1}$$

$$f(x+y) = f(x)f(y), \ (exponential) \tag{1.2}$$

$$f(xy) = f(x) + f(y), \ (logarithmic) \tag{1.3}$$

$$f(xy) = f(x)f(y), \ (multiplicative). \tag{1.4}$$

where f is a real function of a real variable.

Pexiderization of the Cauchy's Equations [10]

One kind of equations with several unknown functions in one variable are pexider's equations. In 1903, J.V. Pexider considered the following functional equations, which are natural generalizations of Cauchy functional equations:

$$f(x+y) = g(x) + h(y),$$

$$f(x+y) = g(x)h(y),$$

$$f(xy) = g(x) + h(y),$$

$$f(xy) = g(x)h(y)$$

for all $x, y \in \mathbb{R}$ with $f, g, h : \mathbb{R} \to \mathbb{R}$.

The functional equation $f(x+y) = f(x) + f(y)$ is the most famous among the functional equations. The properties of the additive Cauchy equation are powerful tools in almost every field of natural and social sciences. Now we first recollect some important facts concerning the Cauchy's additive equations:

Definition 1.2.4. [10] A function $f : \mathbb{R} \to \mathbb{R}$ is said to be rationally homogeneous if and only if

$$f(rx) = rf(x) \tag{1.5}$$

for all $x \in \mathbb{R}$ and all rational numbers r.

Theorem 1.2.5. *[10] Let $f : \mathbb{R} \to \mathbb{R}$ be a solution of the additive Cauchy equation. Then f is rationally homogeneous.*

The solution of additive Cauchy's equation

(i) (**Particular solution**)[12] The particular solution of the additive Cauchy's equation is the following function

$$f(x) = 3x.$$

(ii) (**General solution**) of additive Cauchy's equation given as the following theorem.

Theorem 1.2.6. *[12] Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function or f is bounded from above or below on an interval of positive length satisfying the additive Cauchy functional equation* (1.1)*. Then f is linear, that is; $f(x) = cx$ where c is an arbitrary constant.*

Hyers-Ulam Stability

Theorem 1.2.7. *[10]* ***(Hyers)*** *Let $f : \mathbb{R} \to \mathbb{R}$ is a real function satisfying*

$$|f(x+y) - f(x) - f(y)| \leq \delta \tag{1.6}$$

for some $\delta > 0$ and for all $x, y \in \mathbb{R}$, then there exists a unique additive function $A : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - A(x)| \leq \delta \tag{1.7}$$

for all $x \in \mathbb{R}$.

Proof. To establish this theorem we have to show that

(i) $\left\{\frac{f(2^n x)}{2^n}\right\}_{n=1}^{\infty}$ is a Cauchy sequence for every fixed $x \in \mathbb{R}$.

(ii) if $A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$, then A is additive on $\mathbb{R}$.

(iii) further A satisfies $|f(x) - A(x)| \leq \delta$, for all $x \in \mathbb{R}$.

(iv) A is unique.

Now to show (i), letting $y = x$ in (1.6), we have

$$|f(2x) - 2f(x)| \leq \delta \tag{1.8}$$

for any $x \in \mathbb{R}$. Replacing x by $2^{k-1}x$ in (1.8), (where k is a positive integer greater than or equal to 1), we obtain

$$|f(2^k x) - 2f(2^{k-1}x)| \leq \delta$$

for all $x \in \mathbb{R}$, and $k = 1, 2, ..., n$, where $n \in \mathbb{N}$. Multiplying both sides of the above inequality by $\frac{1}{2^k}$ and adding the resulting n inequalities, we have

$$\sum_{k=1}^{n} \frac{1}{2^k}|f(2^k x) - 2f(2^{k-1}x)| \leq \sum_{k=1}^{n} \frac{1}{2^k}\delta \tag{1.9}$$

since $\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}$, which yields

$$\sum_{k=1}^{n} \frac{1}{2^k}|f(2^k x) - 2f(2^{k-1}x)| \leq \delta \left(1 - \frac{1}{2^n}\right). \tag{1.10}$$

And by using (1.10), we obtain

$$\begin{aligned} |\frac{1}{2^n} f(2^n x) - f(x)| &= |\sum_{k=1}^{n} \frac{1}{2^k}[f(2^k x) - 2f(\sqrt{2}^{k-1}x)]| \\ &\leq \sum_{k=1}^{n} \frac{1}{2^k}|[f(2^k x) - 2f(2^{k-1}x)]| \\ &\leq \delta \left(1 - \frac{1}{2^n}\right) \end{aligned}$$

$$|\frac{1}{2^n} f(2^n x) - f(x)| \leq \delta \left(1 - \frac{1}{2^n}\right) \tag{1.11}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Using induction to show that (1.11) holds for all positive integers $n \in \mathbb{N}$ as the following
since $|\frac{1}{2} f(2x) - f(x)| = \frac{1}{2}|f(2x) - 2f(x)| \leq \frac{\delta}{2}$ by (1.8).
Hence (1.11) is true for $n = 1$.
Suppose that (1.11) is true for $n = k$, we get

$$|\frac{1}{2^k} f(2^k x) - f(x)| \leq \delta \left(1 - \frac{1}{2^k}\right) \tag{a}$$

we show that (1.11) is true for $n = k + 1$.

$$\begin{aligned} |\frac{1}{2^{k+1}} f(2^{k+1} x) - f(x)| &= \frac{1}{2}|\frac{1}{2^k} f(2^k y) - 2f(\frac{y}{2})|, \quad \textit{let } 2x = y,\ x = \frac{y}{2} \\ &= \frac{1}{2}|\frac{1}{2^k} f(2^k y) + f(y) - f(y) - 2f(\frac{y}{2})| \\ &\leq \frac{1}{2}\left[|\frac{1}{2^k} f(2^k y) - f(y)| + |f(y) - 2f(\frac{y}{2})|\right] \\ &\leq \frac{1}{2}\left[\delta(1 - \frac{1}{2^k}) + \delta\right], \ \textit{by } (a) \textit{ and } (1.8) \\ &= \frac{\delta}{2}(1 - \frac{1}{2^k} + 1) \\ &= \frac{\delta}{2}(2 - \frac{1}{2^k}) \\ &= \delta(1 - \frac{1}{2^{k+1}}) \end{aligned}$$

Now if $n > m > 0$, then $n-m$ is a natural number, and n can be replaced by $n-m$ in (1.11) to obtain

$$|\frac{f(2^{n-m}x)}{2^{n-m}} - f(x)| \leq \delta\left(1 - \frac{1}{2^{n-m}}\right)$$

Multiplying both sides by $\frac{1}{2^m}$ and simplifying, we get

$$\left|\frac{f(2^{n-m}x)}{2^n} - \frac{f(x)}{2^m}\right| \leq \frac{\delta}{2^m}\left(1 - \frac{1}{2^{n-m}}\right)$$

for all $x \in \mathbb{R}$. Now we replace x by $2^m x$ in above equation to have

$$\left|\frac{f(2^{n-m}\ 2^m x)}{2^n} - \frac{f(2^m x)}{2^m}\right| \leq \delta\left(\frac{1}{2^m} - \frac{1}{2^n}\right)$$

$$\left|\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}\right| \leq \delta\left(\frac{1}{2^m} - \frac{1}{2^n}\right).$$

If $m \to \infty$, then

$$\left(\frac{1}{2^m} - \frac{1}{2^n}\right) \to 0$$

and therefore

$$\lim_{m\to\infty}\left|\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}\right| = 0.$$

Hence

$$\left\{\frac{f(2^n x)}{2^n}\right\}_{n=1}^{\infty}$$

is a Cauchy sequence in $\mathbb{R}$. Hence the limit of this sequence exists. Define $A : \mathbb{R} \to \mathbb{R}$ by

$$A(x) = \lim_{n\to\infty}\frac{f(2^n x)}{2^n}. \tag{1.12}$$

(ii) Now we show that $A : \mathbb{R} \to \mathbb{R}$ defined by (1.12) is additive.

Consider

$$\begin{aligned}
|A(x+y) - A(x) - A(y)| &= \left|\lim_{n\to\infty}\left\{\frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n}\right\}\right| \\
&= \left|\lim_{n\to\infty}\frac{1}{2^n}\{f(2^n(x+y)) - f(2^n x) - f(2^n y)\}\right| \\
&= \lim_{n\to\infty}\frac{1}{2^n}|f(2^n x + 2^n y) - f(2^n x) - f(2^n y)| \\
&\leq \lim_{n\to\infty}\frac{\delta}{2^n} \quad \textit{by } (1.6) \\
&= 0.
\end{aligned}$$

Therefore

$$A(x+y) = A(x) + A(y)$$

for all $x, y \in \mathbb{R}$.

(iii) Our next goal is to show that

$$|A(x) - f(x)| \leq \delta.$$

Thus consider

$$\begin{aligned}
|A(x) - f(x)| &= \left| \lim_{n\to\infty} \frac{f(2^n x)}{2^n} - f(x) \right| \\
&= \lim_{n\to\infty} \left| \frac{f(2^n x)}{2^n} - f(x) \right| \\
&\leq \lim_{n\to\infty} \delta \left(1 - \frac{1}{2^n}\right) \qquad \textit{by} \;\; (1.11) \\
&= \lim_{n\to\infty} (\delta - \frac{\delta}{2^n}) \\
&= \delta.
\end{aligned}$$

Hence we obtain

$$|A(x) - f(x)| \leq \delta$$

for all $x \in \mathbb{R}$.

(iv) Finally we prove that A is unique. Suppose A is not unique, then there exists another additive function $B : \mathbb{R} \to \mathbb{R}$ such that

$$|B(x) - f(x)| \leq \delta \tag{1.13}$$

for all $x \in \mathbb{R}$. Note that

$$\begin{aligned}
|B(x) - A(x)| &= |B(x) - f(x) + f(x) - A(x)| \\
&\leq |B(x) - f(x)| + |f(x) - A(x)| \\
&= \delta + \delta.
\end{aligned}$$

Therefore

$$|B(x) - A(x)| \leq 2\delta. \tag{1.14}$$

Further, since A and B are additive, and by (1.5) we have

$$\begin{aligned}
|A(x) - B(x)| &= \left| \frac{nA(x)}{n} - \frac{nB(x)}{n} \right| \\
&= \left| \frac{A(nx)}{n} - \frac{B(nx)}{n} \right| \\
&= \frac{1}{n} |A(nx) - B(nx)| \\
&\leq \frac{2\delta}{n}, \qquad \textit{by} \; (1.14)
\end{aligned}$$

where $n \in \mathbb{N}$.

Hence

$$|A(x) - B(x)| \leq \frac{2\delta}{n}.$$

Taking the limit on both sides, we get

$$\lim_{n\to\infty} |A(x) - B(x)| \leq \lim_{n\to\infty} \frac{2\delta}{n}$$

which is

$$|A(x) - B(x)| \leq 0.$$

Hence

$$A(x) = B(x) \qquad \forall x \in \mathbb{R}.$$

Therefore the additive map A is unique and the proof of the theorem is now complete.

□

Remark 1.2.8. [10] Any result similar to Theorem (1.2.7) is known as the Hyers-Ulam stability of the corresponding functional equation.

Remark 1.2.9. [10] In general the proof of Theorem (1.2.7) works for functions $f : E_1 \to E_2$ where E_1 and E_2 are Banach spaces.

Generalizations of Hyers' Theorem

In the following theorem, we present Rassias' result that generated a lot of activities in the stability theory of functional equations.

Theorem 1.2.10. *[13] **(Rassias)** Let E_1 and E_2 be Banach spaces, and let $f : E_1 \to E_2$ be a function satisfying the functional inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\left(\|x\|^p + \|y\|^p\right) \tag{1.15}$$

for some $\theta > 0$, $p \in [0,1)$, and for all $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p \tag{1.16}$$

for any $x \in E_1$.

Remark 1.2.11. [10] Theorem (1.2.10) holds for all $p \in \mathbb{R}\backslash\{1\}$. Gajda (1991) gave an example to show that Theorem (1.2.10) fails if $p = 1$. Gajda succeeded in constructing an example of a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$|f(x+y) - f(x) - f(y)| \leq |x| + |y|$$

for all $x, y \in \mathbb{R}$, with

$$\lim_{x \to 0} \frac{f(x)}{x} = \infty.$$

The function f which Gajda (1991) constructed is the following. For a fixed $\theta > 0$, and $\mu = (1/6)\theta$ define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} 2^{-n} \phi(2^n x), \qquad x \in \mathbb{R},$$

where the function $\phi : \mathbb{R} \to \mathbb{R}$ is given by

$$\phi(x) = \begin{cases} \mu & \text{for } x \in [1, \infty), \\ \mu x & \text{for } x \in (-1, 1), \\ -\mu & \text{for } x \in (-\infty, -1]. \end{cases}$$

This construction shows that Theorem (1.2.10) is false for $p = 1$, as we see in the following theorem.

Theorem 1.2.12. *[13]**(Gajda)** The function f defined above satisfies*

$$|f(x + y) - f(x) - f(y)| \leq \theta(|x| + |y|) \tag{1.17}$$

for all $x, y \in \mathbb{R}$, while there is no constant $\delta \geq 0$ and no additive function $A : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$|f(x) - A(x)| \leq \delta|x| \tag{1.18}$$

for all $x, y \in \mathbb{R}$.

Proof. If $x = y = 0$, then (1.17) is trivially satisfied.

Now, we assume that $0 < |x| + |y| < 1$. Then there exists an $N \in \mathbb{N}$ such that

$$2^{-N} \leq |x| + |y| < 2^{-(N-1)}.$$

Since $|x| \leq |x| + |y| < 2^{-(N-1)}$, then $|2^{N-1}x| < |2^{N-1}.2^{-N+1}| = |2^0| = 1$. Hence $|2^{N-1}x| < 1$, and similarly as above we have $|2^{N-1}y| < 1$ and $|2^{N-1}(x + y)| < 1$, which implies that for each $n \in \{0, 1, ..., N-1\}$ the numbers $2^n x$, $2^n y$ and $2^n(x+y)$ belong to the interval $(-1, 1)$. Since ϕ is linear on this interval, we infer that

$$\phi(2^n(x + y)) = \phi(2^n x) + \phi(2^n y)$$

then,

$$\phi(2^n(x + y)) - \phi(2^n x) - \phi(2^n y) = 0$$

for $n \in \{0, 1, ..., N-1\}$. As a result, we get

$$\begin{aligned}
\frac{|f(x+y) - f(x) - f(y)|}{|x|+|y|} &= \frac{|\sum_{n=0}^{\infty} 2^{-n}\phi(2^n(x+y)) - 2^{-n}\phi(2^n x) - 2^{-n}\phi(2^n y)|}{|x|+|y|} \\
&\leq \sum_{n=0}^{\infty} \frac{|\phi(2^n(x+y)) - \phi(2^n x) - \phi(2^n y)|}{2^n(|x|+|y|)} \\
&\leq \sum_{n=N}^{\infty} \frac{|\phi(2^n(x+y)) - \phi(2^n x) - \phi(2^n y)|}{2^n(|x|+|y|)}
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{n=N}^{\infty} \frac{|\phi(2^n(x+y))| + |\phi(2^n x)| + |\phi(2^n y)|}{2^n(|x|+|y|)} \\
&= \sum_{n=N}^{\infty} \frac{\mu+\mu+\mu}{2^n(|x|+|y|)} \\
&\leq \sum_{k=0}^{\infty} \frac{3\mu}{2^k 2^N(|x|+|y|)} \\
&\leq \sum_{k=0}^{\infty} \frac{3\mu}{2^k} \\
&= 3\mu \sum_{k=0}^{\infty} \frac{1}{2^k} \\
&= 6\mu \\
&= 6(\frac{1}{6}\theta) \\
&= \theta.
\end{aligned}$$

Finally, assume $|x| + |y| \geq 1$. Then merely by means of the boundedness of f and since

$$|f(x)| \leq \sum_{n=0}^{\infty} 2^{-n}\mu = 2\mu$$

we have

$$\begin{aligned}
\frac{|f(x+y) - f(x) - f(y)|}{|x|+|y|} &\leq \frac{|f(x+y)| + |f(x)| + |f(y)|}{|x|+|y|} \\
&= \frac{2\mu+2\mu+2\mu}{|x|+|y|} \\
&= \frac{6\mu}{|x|+|y|} \\
&\leq 6\mu \\
&= \theta.
\end{aligned}$$

Now, contrary to what we claim, suppose that there exist a constant $\delta \geq 0$ and an additive function $A : \mathbb{R} \to \mathbb{R}$ such that (1.18) holds true. Since f is defined by means of a uniformly convergent series of continuous functions, f itself is continuous.

Hence, A is bounded on some neighborhood of zero. Then, by Theorem (1.2.6) there exists a real constant c such that $A(x) = cx$ for all $x \in \mathbb{R}$. Hence, it follows from (1.18) that

$$|f(x) - cx| \leq \delta|x|,$$

for any $x \in \mathbb{R}$, which implies

$$\begin{aligned} |f(x)| - |c||x| &= |f(x)| - |cx| \\ &\leq |f(x) - cx| \\ &\leq \delta|x| \end{aligned}$$

so,

$$\frac{|f(x)|}{|x|} \leq \delta + |c|$$

for all $x \in \mathbb{R}$. On the other hand, we can choose an $N \in \mathbb{N}$ so large that $N\mu > \delta + |c|$. If we choose an $x \in (0, 2^{-(N-1)})$, then we have

$$0 < x < \frac{1}{2^{N-1}},$$

$$0 < 2^n x < \frac{1}{2^{N-n-1}},$$

since $0 \leq n \leq N-1$. Hence $2^n x \in (0,1)$ for each $n \in \{0, 1, ..., N-1\}$. Consequently, for such an x we get

$$\frac{f(x)}{x} = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n x} \geq \sum_{n=0}^{N-1} \frac{\phi(2^n x)}{2^n x} = \sum_{n=0}^{N-1} \frac{\mu 2^n x}{2^n x} = N\mu > \delta + |c|,$$

which leads to a contradiction. $\square$

1.2.2 Quadratic functional equation

Definition 1.2.13. [10] A mapping $f : \mathbb{R} \to \mathbb{R}$ is called a **quadratic functional equation** if

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.19}$$

holds for all $x, y \in \mathbb{R}$.

And every solution of the equation (1.19) is said to be a quadratic mapping.

Theorem 1.2.14. *[10] Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in \mathbb{R}$. Then f is rationally homogeneous of degree 2.
or

$$f(rx) = r^2 f(x)$$

for all $x \in \mathbb{R}$ and all rational numbers r.

The general continuous solution

Theorem 1.2.15. *[10] The general continuous solution of*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y),$$

for all $x, y \in \mathbb{R}$, is given by

$$f(x) = cx^2,$$

where c is an arbitrary constant.

Pexiderization of Quadratic Equation[10]

The quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y),$$

can be pexiderized to

$$f_1(x+y) + f_2(x-y) = f_3(x) + f_4(y),$$

where $f_1, f_2, f_3, f_4 : \mathbb{R} \to \mathbb{R}$ are unknown functions and x, y are real numbers.

Hyers-Ulam Stability

Theorem 1.2.16. *[10] If the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the inequality*

$$|f(x+y) + f(x-y) - 2f(x) - 2f(y)| \leq \delta \tag{1.20}$$

for some $\delta > 0$ and for all $x, y \in \mathbb{R}$, then there exists a unique quadratic function $q : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - q(x)| \leq \frac{\delta}{2}$$

for all $x \in \mathbb{R}$.

Proof. Letting $x = 0 = y$ in (1.20), we see that

$$|f(0)| \leq \frac{\delta}{2}. \tag{1.21}$$

Further, letting $y = x$ in (1.20), we see that

$$|f(2x) + f(0) - 4f(x)| \leq \delta,$$

that is,

$$|f(2x) - 4f(x)| - |f(0)| \leq \delta$$

and by (1.21), we have

$$|f(2x) - 4f(x)| \leq \frac{3}{2}\delta \tag{1.22}$$

for all $x \in \mathbb{R}$. We replacing x by $2^{k-1}x$ in (1.22) to get

$$|f(2^k x) - 2^2 f(2^{k-1}x)| \leq \frac{3}{2}\delta.$$

Multiplying both sides of the last inequality by $\frac{1}{2^{2k}}$ and then summing both sides of the resulting inequality as k goes from 1 to n, we get

$$\sum_{k=1}^{n} \frac{1}{2^{2k}} |f(2^k x) - 2^2 f(2^{k-1}x)| \leq \sum_{k=1}^{n} \frac{3}{2}\frac{1}{2^{2k}}\delta. \tag{1.23}$$

Using the inequality

$$|x| - |y| \leq |x - y|,$$

we see that

$$|\frac{1}{2^{2n}} f(2^n x) - f(x)| \leq \frac{\delta}{2}$$

for all positive integers n.

If $m > n > 0$, then $m - n$ is a natural number and n can be replaced by $m - n$ in the above inequality. Thus, we have

$$\left| \frac{1}{2^{2(m-n)}} f(2^{m-n}x) - f(x) \right| \leq \frac{\delta}{2}$$

or

$$\left|\frac{1}{2^{2m}}f(2^{m-n}x) - \frac{1}{2^{2n}}f(x)\right| \leq \frac{\delta}{2.2^{2n}}.$$

Replacing x by $2^n x$, we get

$$\left|\frac{1}{2^{2m}}f(2^{m}x) - \frac{1}{2^{2n}}f(2^n x)\right| \leq \frac{\delta}{2^{(2n+1)}}.$$

Letting $n \to \infty$, we see that

$$\lim_{n\to\infty} \frac{\delta}{2^{2n+1}} = 0,$$

and hence

$$\left|\frac{1}{2^{2m}}f(2^{m}x) - \frac{1}{2^{2n}}f(2^n x)\right| \to 0 \qquad as\ n \to \infty.$$

Thus

$$\left\{\frac{f(2^n x)}{2^{2n}}\right\}_{n=1}^{\infty}$$

is a Cauchy sequence. Hence this sequence has a limit in $\mathbb{R}$. We define a function $q : \mathbb{R} \to \mathbb{R}$ using this limit by

$$q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{2n}} \tag{1.24}$$

for $x \in \mathbb{R}$. Next, we show that $q(x)$ is quadratic. Since

$$\begin{aligned}
&|q(x+y) + q(x-y) - 2q(x) - 2q(y)| \\
&\quad = \lim_{n\to\infty} \frac{1}{2^{2n}} |f(2^n x + 2^n y) + f(2^n x - 2^n y) - 2f(2^n x) - 2f(2^n y)| \\
&\quad = \lim_{n\to\infty} \frac{\delta}{2^{2n}} = 0. \qquad by\ (1.20),
\end{aligned}$$

therefore, we have

$$q(x+y) + q(x-y) = 2q(x) + 2q(y)$$

for all $x, y \in \mathbb{R}$. Hence q is a quadratic function.

Next, we consider

$$\begin{aligned}
|q(x) - f(x)| &= \left|\lim_{n\to\infty} \frac{f(2^n x)}{2^{2n}} - f(x)\right| \\
&= \lim_{n\to\infty} \left|\frac{f(2^n x)}{2^{2n}} - f(x)\right| \\
&\leq \lim_{n\to\infty} \frac{\delta}{2} \\
&= \frac{\delta}{2}.
\end{aligned}$$

Hence $|q(x) - f(x)| \leq \frac{\delta}{2}$ for all $x \in \mathbb{R}$.

Finally, we prove that q is unique. Suppose $q : \mathbb{R} \to \mathbb{R}$ is not unique. Then there exists another quadratic function $s : \mathbb{R} \to \mathbb{R}$ such that

$$|s(x) - f(x)| \leq \frac{\delta}{2}$$

for all $x \in \mathbb{R}$. Note that

$$\begin{aligned} |s(x) - q(x)| &\leq |s(x) - f(x)| + |f(x) - q(x)| \\ &\leq \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta. \end{aligned}$$

Therefore

$$|s(x) - q(x)| \leq \delta \tag{1.25}$$

for all $x \in \mathbb{R}$.

Since a quadratic function is rationally homogeneous of degree two, we have

$$\begin{aligned} |s(x) - q(x)| &= \left| \frac{n^2 s(x)}{n^2} - \frac{n^2 q(x)}{n^2} \right| \\ &= \left| \frac{s(nx)}{n^2} - \frac{q(nx)}{n^2} \right| \\ &= \frac{1}{n^2} |s(nx) - q(nx)| \\ &\leq \frac{\delta}{n^2}. \end{aligned}$$

Taking the limit $n \to \infty$, we get

$$\lim_{n\to\infty} |s(x) - q(x)| \leq \lim_{n\to\infty} \frac{\delta}{n^2}.$$

Hence

$$|s(x) - q(x)| \leq 0.$$

Therefore

$$s(x) = q(x)$$

for all $x \in \mathbb{R}$. Therefore q is unique. This completes the proof. $\square$

Remarks 1.2.17. [10]

1. The proof of the above theorem goes over without any changes if one replaces the real function $f : \mathbb{R} \to \mathbb{R}$ by a function from a normed space into a Banach space.

2. In Theorem (1.2.16) the parameter p is assumed to take all values except 2. If $p = 2$, then Theorem (1.2.16) is no longer valid. Czerwik (1992) proved the following theorem.

Theorem 1.2.18. *[13] Let us define a function $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(x) = \sum_{n=0}^{\infty} 4^{-n} \phi(2^n x),$$

where the function $\phi : \mathbb{R} \to \mathbb{R}$ *is given by*

$$\phi(x) = \begin{cases} a & \textit{for } |x| \geq 1, \\ ax^2 & \textit{for } |x| < 1. \end{cases}$$

with a positive number a*. The function* f *satisfies the inequality*

$$|f(x+y) + f(x-y) - 2f(x) - 2f(y)| \leq 32a(x^2 + y^2) \tag{1.26}$$

for all $x, y \in \mathbb{R}$*. Moreover, there exists no quadratic function* $Q : \mathbb{R} \to \mathbb{R}$ *such that the image set of* $|f(x) - Q(x)|/x^2$ $(x \neq 0)$ *is bounded.*

Proof. First we show that f is bounded by $(4/3)a$, by definition of f we have

$$f(x) = \begin{cases} \sum_{n=0}^{\infty} 4^{-n} a & \text{if } |2^n x| \geq 1, \\ \sum_{n=0}^{\infty} 4^{-n} a (2^n x)^2 & \text{if } |2^n x| < 1. \end{cases}$$

Then

$$f(x) = \sum_{n=0}^{\infty} a.4^{-n} = a\left(\frac{1}{1 - \frac{1}{4}}\right) = a\frac{1}{3/4} = a\frac{4}{3} \qquad if\ |2^n x| \geq 1,$$

and

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a.4^{-n}.(2^n x)^2 = \sum_{n=0}^{\infty} a.4^{-n}.4^n.x^2 = a\sum_{n=0}^{\infty} x^2 \\ &< a\sum_{n=0}^{\infty} \frac{1}{(2^n)^2} = a\sum_{n=0}^{\infty} \frac{1}{4^n} = a\frac{4}{3} \qquad if\ |x| < \frac{1}{2^n}. \end{aligned}$$

Hence $|f(x)| \leq \frac{4}{3}a$.

For $x = y = 0$ or for $x, y \in \mathbb{R}$ such that $x^2 + y^2 \geq 1/4$, it is clear that the inequality (1.26) holds true because f is bounded by $(4/3)a$ as the following

$$\begin{aligned} |f(x+y) + f(x-y) - 2f(x) - 2f(y)| &\leq |f(x+y)| + |f(x-y)| + |2f(x)| + |2f(y)| \\ &\leq (\frac{4}{3})a + (\frac{4}{3})a + 2(\frac{4}{3})a + 2(\frac{4}{3})a \\ &= (\frac{24}{3})a \\ &= 8a = \frac{32}{4}a \leq 32a(x^2 + y^2), \end{aligned}$$

since $\frac{1}{4} \leq x^2 + y^2$.

Consider the case $0 < x^2 + y^2 < 1/4$. Then there exists an $k \in \mathbb{N}$ such that

$$4^{-k-1} \leq x^2 + y^2 < 4^{-k}, \tag{a}$$

since $x^2 < x^2 + y^2 < 4^{-k}$ then $4^{k-1}x^2 < 4^{k-1}.4^{-k} = 4^{k-1-k} = 4^{-1}$. Hence $4^{k-1}x^2 < 1/4$. and similarly $4^{k-1}y^2 < 1/4$ and consequently

$$2^{k-1}x,\ 2^{k-1}y,\ 2^{k-1}(x+y),\ 2^{k-1}(x-y) \in (-1, 1).$$

Therefore, for each $n \in \{0, 1, ..., k-1\}$, we have

$$2^n x,\ 2^n y,\ 2^n(x+y),\ 2^n(x-y) \in (-1, 1)$$

and

$$\phi(2^n(x+y)) + \phi(2^n(x-y)) - 2\phi(2^n x) - 2\phi(2^n y) = 0$$

for $n \in \{0, 1, ..., k-1\}$. Using (a) we obtain

$$\begin{aligned}
&|f(x+y) + f(x-y) - 2f(x) - 2f(y)| \\
&= |\sum_{n=0}^{\infty} 4^{-n}\phi(2^n(x+y)) + 4^{-n}\phi(2^n(x-y)) - 2.4^{-n}\phi(2^n x) - 2.4^{-n}\phi(2^n y)| \\
&\leq \sum_{n=0}^{\infty} 4^{-n}|\phi(2^n(x+y)) + \phi(2^n(x-y)) - 2\phi(2^n x) - 2\phi(2^n y)| \\
&\leq \sum_{n=0}^{\infty} 4^{-n}|\phi(2^n(x+y))| + |\phi(2^n(x-y))| + |2\phi(2^n x)| + |2\phi(2^n y| \\
&= \sum_{n=0}^{\infty} 4^{-n}(a + a + 2a + 2a) \\
&\leq \sum_{n=k}^{\infty} 4^{-n}6a = 6a.4^{-k}.\frac{1}{1-\frac{1}{4}} = \frac{6a.4^{1-k}}{3} = 2a.4^{1-k} = 2a.4.4^{-k}(4.4^{-1}) = 32a.4^{-k-1} \\
&\leq 32a(x^2 + y^2).
\end{aligned}$$

i.e. the inequality (1.26) holds true.

Assume that there exist a quadratic function $Q : \mathbb{R} \to \mathbb{R}$ and a constant $b > 0$ such that

$$|f(x) - Q(x)| \leq bx^2$$

for all $x \in \mathbb{R}$. Since Q is locally bounded, it is of the form $Q(x) = cx^2$ $(x \in \mathbb{R})$, where c is constant. Therefor, we have

$$|f(x)| - |cx^2| \leq |f(x) - cx^2| \leq bx^2$$

then

$$|f(x)| \leq bx^2 + |c|x^2 = (b + |c|)x^2$$

hence

$$|f(x)| \leq (b + |c|)x^2 \qquad (b)$$

for all $x \in \mathbb{R}$. Let $k \in \mathbb{N}$ satisfy $ka > b + |c|$. If $x \in (0, 2^{1-k})$, we have

$$0 < x < 2^{1-k}$$

and

$$0 < 2^n x < 2^n.2^{-(k-1)}$$
$$0 < 2^n x < \frac{2^n}{2^{k-1}}$$

since $0 \leq n \leq k-1$, then $2^n x \in (0,1)$ and we have

$$f(x) = \sum_{n=0}^{\infty} 4^{-n}\phi(2^n x) \geq \sum_{n=0}^{k-1} a4^{-n}(2^n x)^2 = kax^2 > (b+|c|)x^2,$$

which in comparison with (b) is a contradiction. $\square$

1.2.3 The Jensen Functional Equation

Definition 1.2.19. [10] A function $f : \mathbb{R} \to \mathbb{R}$ is said to be convex if and only if it satisfies the inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

for all $x, y \in \mathbb{R}$.

Example 1.2.20. *[10] The followings are examples of convex functions:*

(1) $f(x) = mx + c$ *on* $\mathbb{R}$*, for any* $m, c \in \mathbb{R}$.

(2) $f(x) = x^2$ *on* $\mathbb{R}$.

(3) $f(x) = x \log x$ *on* $\mathbb{R}_+$.

(4) $f(x) = |x|^\alpha$ *on* $\mathbb{R}$ *for any* $\alpha \geq 1$.

Definition 1.2.21. [10] A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **Jensen** or Jensen additive if it satisfies

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}, \qquad (JE)$$

for all $x, y \in \mathbb{R}$. Every solution of Jensens functional equation is called a **Jensen function**, and we can refer to above equation by (JE).

Definition 1.2.22. [1] A mapping $f : X \to Y$ is called a **Jensen quadratic function** if it satisfies:

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y),$$

for all $x, y \in X$. Where X is a normed space and Y is a Banach space.

The solution of the additive Jensen functional equation

Theorem 1.2.23. *[2] The most general continuous solution of* (JE) *in all* $\mathbb{R}$ *is*

$$f(x) = cx + a,$$

where c *and* a *are arbitrary constants.*

Pexiderization of the Jensen Functional Equation:[10]

The Jensen functional equation can be generalized to

$$f\left(\frac{x+y}{2}\right)=\frac{g(x)+h(y)}{2} \qquad for\ x,y\in\mathbb{R},$$

where $f,g,h:\mathbb{R}\to\mathbb{R}$ are unknown functions to be determined.

HyersUlam Stability

Theorem 1.2.24. *[13] Let E_1 and E_2 be a real normed space and a real Banach space, respectively. Assume that $\delta,\ \theta\geq 0$ are fixed, and let $p>0$ be given with $p\neq 1$. Suppose a function $f:E_1\to E_2$ satisfies the functional inequality*

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\|\leq\theta(\parallel x\parallel^p+\parallel y\parallel^p)$$

for all $x,\ y\in E_1$. Then there exists a unique additive function $A:E_1\to E_2$ such that

$$\parallel f(x)-A(x)\parallel\leq\begin{cases}(2^{1-p}-1)^{-1}\ \theta\parallel x\parallel^p & for\ \ p<1,\\ 2^{p-1}(2^{p-1}-1)^{-1}\ \theta\parallel x\parallel^p & for\ \ p>1.\end{cases}$$

for all $x\in E_1$.

The proof of the stability of the Jensen functional equation here is similar to that demonstrated in the following section, which prove the stability of the Jensen functional equation in the orthogonality space.

Remark 1.2.25. [13] As discussed at the end of Theorem (1.2.10). that the functional inequality

$$||f(x+y)-f(x)-f(y)||\leq\theta(||x||+||y||)$$

is not stable. By using this result, S-M.Jung proved that the function constructed by Rassias serves as a counterexample to Theorem (1.2.24) for the case $p=1$ as follows:

Theorem 1.2.26. *[13] The continuous real-valued function defined by*

$$f(x)=\begin{cases}x\log_2(x+1) & for\ x\geq 0,\\ x\log_2|x-1| & for\ x<0.\end{cases}$$

satisfies the inequality

$$\left|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right|\leq 2(|x|+|y|), \tag{1.27}$$

for all $x,y\in\mathbb{R}$, and the image set of $|f(x)-A(x)|/|x|$ for $x\neq 0$ is unbounded for each additive function $A:\mathbb{R}\to\mathbb{R}$.

Proof. The given function f is continuous, odd, and convex on $(0,\infty)$. Let x and y be positive numbers. Since f is convex on $(0,\infty)$, it follows from the fact

$$\begin{aligned}|f(x+y)-f(x)-f(y)| &= f(x+y)-f(x)-f(y)\\ &\leq f(x+y)-2f\left(\frac{x+y}{2}\right)\end{aligned}$$

then

$$|f(x+y)-f(x)-f(y)| \leq f(x+y)-2f\left(\frac{x+y}{2}\right) \qquad (a)$$

that

$$\begin{aligned}|f(x+y)-f(x)-f(y)| &\leq f(x+y)-2f\left(\frac{x+y}{2}\right)\\ &= (x+y)\log_2(x+y+1)-2\left(\frac{x+y}{2}\right)\log_2\left(\frac{x+y}{2}+1\right)\\ &= (x+y)\log_2(x+y+1)-(x+y)\log_2\left(\frac{x+y+2}{2}\right)\\ &= (x+y)\left(\log_2\left(\frac{x+y+1}{\frac{x+y+2}{2}}\right)\right)\\ &= (x+y)\log_2\frac{2x+2y+2}{x+y+2}\\ &< |x|+|y|\end{aligned}$$

hence

$$|f(x+y)-f(x)-f(y)| \leq (x+y)\log_2\frac{2+2x+2y}{2+x+y} < |x|+|y| \qquad (b)$$

for all $x,y>0$. Since f is odd function, (b) holds true for $x,y<0$ as the following

$$\begin{aligned}|f(x+y)-f(x)-f(y)| &= |f(x)-f(x+y)-f(-y)|\\ &\leq f(x)-2f(\frac{x+y-y}{2})\\ &= f(x)-2f(x/2)\\ &= x\log_2|x-1|-2\frac{x}{2}\log_2\left|\frac{x}{2}-1\right|\\ &= x\log_2|x-1|-x\log_2\left|\frac{x-2}{2}\right|\\ &= x(\log_2\frac{|x-1|}{\left|\frac{x-2}{2}\right|})\\ &= x\log_2\frac{|2x-2|}{|x-2|}\\ &< |x|+|y|,\end{aligned}$$

also (b) hold for $x = 0,\ y = 0$ or $x + y = 0$ as following

$$\begin{aligned}
|f(x+y) - f(x) - f(y)| &\le f(x+y) - 2f(\frac{x+y}{2}) \\
&= (x+y)\log_2(x+y+1) - 2(\frac{x+y}{2})\log_2(\frac{x+y+1}{2}) \\
&= (x+y)\log_2(\frac{2x+2y+2}{x+y+1}) \\
&= 0 = |x+y| \\
&\le |x| + |y|,
\end{aligned}$$

hence (b) holds true for $x = 0,\ y = 0$, or $x + y = 0$, it only remains to consider the case when $x > 0$ and $y < 0$. Without loss of generality, assume $|x| > |y|$. By oddness and convexity of f and by (a), we get

$$\begin{aligned}
|f(x+y) - f(x) - f(y)| &= |f(x) - f(x+y) - f(-y)| \\
&\le f(x) - 2f(\frac{x+y-y}{2}) \\
&= f(x) - 2f(x/2) \\
&= x\log_2(x+1) - 2(x/2)\log_2(\frac{x}{2}+1) \\
&= x\log_2(x+1) - x\log_2(\frac{x+2}{2}) \\
&= x(\log_2(\frac{x+1}{\frac{x+2}{2}})) \\
&= x\log_2(\frac{2x+2}{x+2}) \\
&< |x| + |y|,
\end{aligned}$$

since $x + y$ and $-y$ are positive numbers. Thus, the inequality (b) holds true for all $x, y \in \mathbb{R}$.

By substituting $x/2$ and $y/2$ for x and y in (b), respectively, and multiplying by 2 both sides, we have

$$\begin{aligned}
2\left|f(\frac{x}{2}+\frac{y}{2}) - f(\frac{x}{2}) - f(\frac{y}{2})\right| &= \left|2f(\frac{x+y}{2}) - 2f(\frac{x}{2}) - 2f(\frac{y}{2})\right| \\
&\le 2\left|\frac{x}{2}\right| + 2\left|\frac{y}{2}\right| \\
&= |x| + |y|,
\end{aligned}$$

hence

$$\left|2f(\frac{x+y}{2}) - 2f(x/2) - 2f(y/2)\right| \le |x| + |y| \qquad (c)$$

for all $x, y \in \mathbb{R}$. Putting $x = y$ and dividing by 2 both sides in (c) yield

$$\begin{aligned}
\frac{1}{2}|2f(\frac{x+x}{2})-2f(\frac{x}{2})-2f(\frac{x}{2})| &= |f(x)-f(\frac{x}{2})-f(\frac{x}{2})| \\
&= |f(x)-2f(\frac{x}{2})| \\
&\leq \frac{1}{2}(|x|+|x|)=\frac{1}{2}(2|x|) \\
&= |x|
\end{aligned}$$

$$|f(x)-2f(x/2)| \leq |x| \qquad (d)$$

for $x \in \mathbb{R}$. By using (c) we obtain

$$\begin{aligned}
&\left|2f(\frac{x+y}{2})-2f(x/2)-2f(y/2)\right| \\
&\quad = \left|2f(\frac{x+y}{2})-f(x)-f(y)+f(x)-2f(x/2)+f(y)-2f(y/2)\right| \\
&\quad \leq |x|+|y|
\end{aligned}$$

for all $x, y \in \mathbb{R}$. The validity of (1.27) follows immediately from (d) and the last inequality as the following

$$\begin{aligned}
&\left|2f(\frac{x+y}{2})-f(x)-f(y)\right| \\
&\quad = \left|2f(\frac{x+y}{2})-2f(x/2)-2f(y/2)-f(x)+2f(x/2)-f(y)+2f(y/2)\right| \\
&\quad \leq |2f(\frac{x+y}{2})-2f(x/2)-2f(y/2)|+|f(x)-2f(x/2)|+|f(y)-2f(y/2)| \\
&\quad \leq (|x|+|y|)+|x|+|y| = 2(|x|+|y|).
\end{aligned}$$

It is well-known that if an additive function $A : \mathbb{R} \to \mathbb{R}$ is continuous at a point, then $A(x) = cx$ where c is a real number by theorem (1.2.6). It is trivial that $|f(x) - cx|/|x| \to \infty$ as $x \to \infty$ for any real number c, and that the image set of $|f(x) - A(x)|/|x|$ for $x \neq 0$ is also unbounded for every additive function $A : \mathbb{R} \to \mathbb{R}$ which is not continuous because the graph of the function A is everywhere dense in $\mathbb{R}^2$. □

1.2.4 Cosine Functional Equation

The trigonometric functions $f(x) = \cos(x)$ and $g(x) = \sin(x)$ satisfy the functional equations

$$f(x+y) = f(x)f(y) - g(x)g(y), \tag{1.28}$$

$$f(x-y) = f(x)f(y) + g(x)g(y) \tag{1.29}$$

for all $x, y \in \mathbb{R}$. Adding (1.28) and (1.29), we get

$$f(x+y) + f(x-y) = 2f(x)f(y) \tag{1.30}$$

where f and g may be defined on a group or semigroup with values in a field K which usually is the field of real or complex numbers. This equation is called the cosine functional equation or **d'Alembert equation**, also we can refer to equation (1.30) by (DE).

In the following theorems, we first present the continuous regular and general solutions of the d'Alembert equation.

The regular solution

Theorem 1.2.27. *[10] Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and satisfy*

$$f(x+y) + f(x-y) = 2f(x)f(y),$$

for all $x, y \in \mathbb{R}$. Then f is of the form

$$f(x) = 0, \quad f(x) = 1, \quad f(x) = cosh(\alpha x), \quad f(x) = cos(\beta x),$$

where α and β are arbitrary real constants.

The general solution

Theorem 1.2.28. *[13] Every nontrivial solution $f : \mathbb{R} \to \mathbb{C}$ of the functional equation*

$$f(x+y) + f(x-y) = 2f(x)f(y),$$

given by

$$f(x) = \frac{m(x) + m(-x)}{2},$$

where $m : \mathbb{R} \to \mathbb{C} \backslash \{0\}$ is an exponential function.

The Pexiderization of the d'Alembert equation [6]

the pexiderized functional equations of the d'Alembert equation as follows:

$$f(x+y) + f(x-y) = 2g(x)h(y), \tag{1.31}$$

$$f(x+y) + g(x-y) = 2f(x)g(y), \tag{1.32}$$

$$f(x+y) + g(x-y) = 2g(x)f(y). \tag{1.33}$$

Where $f,\ g,\ h : \mathbb{R} \to \mathbb{C}$ for above equation.
We will focus in chapter three on the stability of the following pexiderized cosine functional equation:

$$f_1(x+y) + f_2(x-y) = 2g_1(x)g_2(y)$$

where f_1, f_2, g_1, and g_2 are functions from $\mathbb{R}$ to $\mathbb{C}$.
we can consider the above equation as a special case of the equation (1.31) where $f_1 = f_2 = f$, $g_1 = g$ and $g_2 = h$.

Lemma 1.2.29. *[13] Let $(G, +)$ be an abelian group and let a function $f : G \to \mathbb{C}$ satisfy the functional inequality*

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \delta$$

for all $x,\ y \in G$ and for some $\delta > 0$. If $|f(x)| > (1 + \sqrt{1+2\delta})/2$ for some $x \in G$, then $|f(2^n x)| \to \infty$ as $n \to \infty$.

Stability of d'Alembert Equation

Theorem 1.2.30. *[13] Let $(G, +)$ be an abelian group and let a function $f : G \to \mathbb{C}$ satisfy the functional inequality*

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \delta \qquad (1.34)$$

for all $x, y \in G$ and for some $\delta > 0$. Then either $|f(x)| \leq (1 + \sqrt{1+2\delta})/2$ for any $x \in G$ or f satisfies the cosine functional equation (1.30) for all $x,\ y \in G$.

Proof. If there exists an $x_o \in G$ such that

$$|f(x_o)| > (1 + \sqrt{1+2\delta})/2,$$

then by above lemma there exists a sequence $\{x_n\}$ with

$$\lim_{n\to\infty} |f(x_n)| = \infty. \qquad (a)$$

Let $x, y \in G$ be given. From (1.34) it follows that

$$|2f(x_n)f(x) - f(x+x_n) - f(x-x_n)| \leq \delta$$

for all $n \in \mathbb{N}$. Which is

$$-\delta \leq 2f(x_n)f(x) - f(x+x_n) - f(x-x_n) \leq \delta.$$

From the inequality we have

$$2f(x_n)f(x) - f(x+x_n) - f(x-x_n) - \delta \leq 0,$$

so

$$2f(x_n)f(x) - f(x+x_n) - f(x-x_n) - \delta = 0,$$

and

$$2f(x_n)f(x) = f(x+x_n) - f(x-x_n).$$

Therefore

$$f(x) = \lim_{n\to\infty} \frac{f(x+x_n) + f(x-x_n)}{2f(x_n)}. \qquad (b)$$

From (b) we get

$$\begin{aligned} 2f(x)f(y) &= 2\lim_{n\to\infty} \frac{(f(x+x_n)+f(x-x_n))(f(y+x_n)+f(y-x_n))}{4f(x_n)^2} \\ &= \lim_{n\to\infty} \frac{(f(x+x_n)+f(x-x_n))(f(y+x_n)+f(y-x_n))}{2f(x_n)^2} \\ &= \lim_{n\to\infty} A_n \end{aligned}$$

hence

$$2f(x)f(y) = \lim_{n\to\infty} A_n \qquad (c)$$

where

$$A_n = \frac{(f(x+x_n)+f(x-x_n))(f(y+x_n)+f(y-x_n))}{2f(x_n)^2}.$$

And

$$\begin{aligned} f(x+y)+f(x-y) &= \lim_{n\to\infty} \frac{f(x+y+x_n)+f(x+y-x_n)+f(x-y+x_n)+f(x-y-x_n)}{2f(x_n)} \\ &= \lim_{n\to\infty} \frac{[f(x+y+x_n)+f(x+y-x_n)}{2f(x_n)^2} \\ &\qquad + \frac{f(x-y+x_n)+f(x-y-x_n)]\, f(x_n)}{2f(x_n)^2} \\ &= \lim_{n\to\infty} B_n \end{aligned}$$

hence

$$f(x+y)+f(x-y) = \lim_{n\to\infty} B_n \qquad (d)$$

where

$$B_n = (1/2)f(x_n)^{-2}\,(f(x+y+x_n)+f(x+y-x_n)$$
$$+f(x-y+x_n)+f(x-y-x_n))\,f(x_n).$$

By (1.34), we have

$$|2f(x+x_n)f(y+x_n) - f(x+y+2x_n) - f(x-y)| \le \delta,$$

$$|2f(x-x_n)f(y+x_n) - f(x+y) - f(x-y-2x_n)| \le \delta,$$

$$|2f(x+x_n)f(y-x_n) - f(x+y) - f(x-y+2x_n)| \le \delta,$$

$$|2f(x-x_n)f(y-x_n) - f(x+y-2x_n) - f(x-y)| \le \delta,$$

$$|-2f(x+y+x_n)f(x_n) + f(x+y+2x_n) + f(x+y)| \le \delta,$$

$$|-2f(x+y-x_n)f(x_n) + f(x+y) + f(x+y-2x_n)| \le \delta,$$

$$|-2f(x-y+x_n)f(x_n) + f(x-y) + f(x-y+2x_n)| \le \delta,$$

$$|-2f(x-y-x_n)f(x_n) + f(x-y) + f(x-y-2x_n)| \le \delta$$

for every $n \in \mathbb{N}$, and hence

$$\begin{aligned}
2|A_n - B_n| = 2\Bigg| &\frac{f(x+x_n)f(y+x_n) + f(x+x_n)f(y-x_n)}{2f(x_n)^2} \\
&+ \frac{f(x-x_n)f(y+x_n) + f(x-x_n)f(y-x_n)}{2f(x_n)^2} \\
&- \frac{f(x+y+x_n)f(x_n) - f(x+y-x_n)f(x_n)}{2f(x_n)^2} \\
&- \frac{f(x-y+x_n)f(x_n) - f(x-y-x_n)f(x_n)}{2f(x_n)^2}\Bigg| \\
\le &\left|\frac{2f(x+x_n)f(y+x_n) - 2f(x+y+x_n)f(x_n)}{2f(x_n)^2}\right| \\
&+ \left|\frac{2f(x+x_n)f(y-x_n) - 2f(x+y-x_n)f(x_n)}{2f(x_n)^2}\right| \\
&+ \left|\frac{2f(x-x_n)f(y+x_n) - 2f(x-y+x_n)f(x_n)}{2f(x_n)^2}\right| \\
&+ \left|\frac{2f(x-x_n)f(y-x_n) - 2f(x-y-x_n)f(x_n)}{2f(x_n)^2}\right|
\end{aligned}$$

$$\begin{aligned}
\le &\frac{|2f(x+x_n)f(y+x_n)| + \delta - |f(x+y+2x_n) + f(x+y)|}{|2f(x_n)^2|} \\
+ &\frac{|2f(x+x_n)f(y-x_n)| + \delta - |f(x+y-2x_n) + f(x+y)|}{|2f(x_n)^2|} \\
+ &\frac{|2f(x-x_n)f(y+x_n)| + \delta - |f(x-y+2x_n) + f(x-y)|}{|2f(x_n)^2|} \\
+ &\frac{|2f(x-x_n)f(y-x_n)| + \delta - |f(x-y-2x_n) + f(x-y)|}{|2f(x_n)^2|}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\delta + |2f(x+x_n)f(y+x_n) - f(x+y+2x_n) - f(x+y)|}{|2f(x_n)^2|} \\
&+ \frac{\delta + |2f(x+x_n)f(y-x_n) - f(x+y-2x_n) - f(x+y)|}{|2f(x_n)^2|} \\
&+ \frac{\delta + |2f(x-x_n)f(y+x_n) - f(x-y+2x_n) - f(x-y)|}{|2f(x_n)^2|} \\
&+ \frac{\delta + |2f(x-x_n)f(y-x_n) - f(x-y-2x_n) - f(x-y)|}{|2f(x_n)^2|} \\
&\leq \frac{8\delta}{2|f(x_n)^2|}.
\end{aligned}
$$

Hence

$$
|A_n - B_n| \leq \frac{2\delta}{|f(x_n)^2|}, \tag{e}
$$

for all $n \in \mathbb{N}$. The relations (a), (c), (d) and (e) imply that f satisfies the (DE) functional equation for any $x, y \in G$. $\square$

Chapter 2

On the Stability of Orthogonal Functional Equations

2.1 Stability of the orthogonally Jensen additive functional equation

This chapter consists of two sections. In the first one we discuss the stability of the orthogonality Jensen additive functional equation, where Park and Rassias get some new results (as we shall see in this section) in [1].
In the second section we continue with Park and Rassias in [1], where we study the stability of the orthogonality Jensen quadratic functional equation, where some new result Park and Rassias will be explained here.

Definition 2.1.1. [1] Let X be a real vector space with $\dim X \geq 2$ and $\perp$ be a binary relation on X with the following properties:

(1) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$.

(2) independence: if $x, y \in X - \{0\}$, $x \perp y$, then x, y are linearly independent.

(3) homogeneity: if $x, y \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$.

(4) the Thalesian property: if P is a 2-dimensional subspace of X, $x \in P$ and $\lambda \in \mathbb{R}_+$, then $\exists\, y_o \in P$ such that $x \perp y_o$ and $x + y_o \perp \lambda x - y_o$.
The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space we mean an orthogonality space equipped with a norm.

Note 2.1.2. *[1]Throughout this chapter,* $(X, \perp)$ *denotes an orthogonality normed space with norm* $\| \,.\, \|_X$ *and* $(Y, \| \,.\, \|_Y)$ *is a Banach space.*

Theorem 2.1.3. *[1] Let θ and p $(p < 1)$ be nonnegative real numbers. Suppose that $f : X \to Y$ is a mapping with $f(0) = 0$ fulfilling*

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\|_Y \leq \theta(\|x\|_X^p + \|y\|_X^p) \tag{2.1}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally Jensen additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\|_Y \leq \frac{2^p\theta}{2-2^p}\|x\|_X^p \tag{2.2}$$

for all $x \in X$.

Proof. Putting $y = 0$ in (2.1), we get

$$\|2f(\frac{x+0}{2}) - f(x) - f(0)\|_Y \leq \theta(\|x\|_X^p + \|0\|_X^p)$$

since $f(0) = 0$, then

$$\|2f(\frac{x}{2}) - f(x) - 0\|_Y \leq \theta(\|x\|_X^p + 0)$$

so

$$\|2f(\frac{x}{2}) - f(x)\|_Y \leq \theta\|x\|_X^p \tag{2.3}$$

for all $x \in X$, with $x \perp 0$. By replacing x by $2x$ in above inequality and dividing both sides by 2, we have

$$\|f(x) - \frac{1}{2}f(2x)\|_Y \leq \frac{2^p\theta}{2}\|x\|_X^p$$

for all $x \in X$. Now let $n,\ m$ be a nonnegative integers with $n < m$, then by above inequality we have

$$\begin{aligned}
&\|\frac{1}{2^n}f(2^n x) - \frac{1}{2^m}f(2^m x)\|_Y \\
&\quad = \|\frac{1}{2^n}f(2^n x) - \frac{1}{2^{n+1}}f(2^{n+1}x) + \frac{1}{2^{n+1}}f(2^{n+1}x) \\
&\qquad -\frac{1}{2^{n+2}}f(2^{n+2}x) + ... + \frac{1}{2^{m-1}}f(2^{m-1}x) - \frac{1}{2^m}f(2^m x)\|_Y \\
&\quad \leq \|\frac{1}{2^n}f(2^n x) - \frac{1}{2^{n+1}}f(2^{n+1}x)\|_Y + \|\frac{1}{2^{n+1}}f(2^{n+1}x) \\
&\qquad -\frac{1}{2^{n+2}}f(2^{n+2}x)\|_Y + ... + \|\frac{1}{2^{m-1}}f(2^{m-1}x) - \frac{1}{2^m}f(2^m x)\|_Y \\
&\quad = \frac{1}{2^n}\|f(2^n x) - \frac{1}{2}f(2^{n+1}x)\|_Y + \frac{1}{2^{n+1}}\|f(2^{n+1}x) \\
&\qquad -\frac{1}{2}f(2^{n+2}x)\|_Y + ... + \frac{1}{2^{m-1}}\|f(2^{m-1}x) - \frac{1}{2}f(2^m x)\|_Y \\
&\quad \leq \frac{1}{2^n}.\frac{2^p\theta}{2}\|2^n x\|_X^p + \frac{1}{2^{n+1}}.\frac{2^p\theta}{2}\|2^{n+1}x\|_X^p + ... + \frac{1}{2^{m-1}}.\frac{2^p\theta}{2}\|2^{m-1}x\|_X^p
\end{aligned}$$

$$= \frac{2^p\theta}{2}\|x\|_X^p\left(\frac{2^{pn}}{2^n}+\frac{2^{p(n+1)}}{2^{n+1}}+......+\frac{2^{p(m-1)}}{2^{m-1}}\right)$$
$$= \frac{2^p\theta}{2}\|x\|_X^p\sum_{k=n}^{m-1}\frac{2^{kp}}{2^k}.$$

Hence

$$\|\frac{1}{2^n}f(2^nx)-\frac{1}{2^m}f(2^mx)\|_Y \leq \frac{2^p\theta}{2}\sum_{k=n}^{m-1}\frac{2^{pk}}{2^k}\|x\|_X^p \tag{2.4}$$

for all nonnegative integers n, m with $n < m$. Thus $\{\frac{1}{2^n}f(2^nx)\}$ is a Cauchy sequence in Y. Since Y is complete, there exists a mapping $T : X \to Y$ defined by

$$T(x) = \lim_{n\to\infty}\frac{1}{2^n}f(2^nx)$$

for all $x \in X$. Letting $n = 0$ and $m \to \infty$ in (2.4), we get the inequality (2.2) as the following

$$\begin{aligned}
\|f(x)-T(x)\|_Y &= \lim_{m\to\infty}\|\frac{1}{2^0}f(2^0x)-\frac{1}{2^m}f(2^mx)\|_Y\\
&\leq \lim_{m\to\infty}\frac{2^p\theta}{2}\sum_{k=0}^{m-1}\frac{2^{pk}}{2^k}\|x\|_X^p\\
&= \frac{2^p\theta}{2}\|x\|_X^p\lim_{m\to\infty}\sum_{k=0}^{m-1}2^{(p-1)k}\ \textit{since}\ p<1\\
&= \frac{2^p\theta}{2}\|x\|_X^p\frac{1}{1-\frac{2^p}{2}} = \frac{2^p\theta}{2}\|x\|_X^p\frac{1}{\frac{2-2^p}{2}}\\
&= \|x\|_X^p\frac{2^p\theta}{2-2^p}.
\end{aligned}$$

In order to show that T is orthogonally Jensen additive, choose arbitrarily $x, y \in X$, $x \perp y$. Then for any $n \in \mathbb{N}$, and from homogeneity of orthogonal relation one has $2^nx \perp 2^ny$, whence

$$\|2T\ (\frac{x+y}{2})-T(x)-T(y)\|_Y = \|2\lim_{n\to\infty}\frac{1}{2^n}f(\frac{2^nx+2^ny}{2})-\lim_{n\to\infty}\frac{1}{2^n}f(2^nx)-\lim_{n\to\infty}\frac{1}{2^n}f(2^ny)\|_Y$$

$$= \lim_{n\to\infty}\frac{1}{2^n}\|2f(\frac{2^nx+2^ny}{2})-f(2^nx)-f(2^ny)\|_Y$$

$$\leq \lim_{n\to\infty}\frac{\theta}{2^n}(\|2^nx\|_X^p+\|2^ny\|_X^p)$$

$$= \theta(\|x\|_X^p+\|y\|_X^p)\lim_{n\to\infty}\frac{2^{pn}}{2^n}$$

$$= \theta(\|x\|_X^p+\|y\|_X^p)\lim_{n\to\infty}2^{(p-1)n} = 0$$

since $|2^{p-1}| < 1$, where $\lim_{n\to\infty} 2^{(p-1)n} = 0$ for all $x,\ y \in X$ with $x \perp y$. So

$$2T(\frac{x+y}{2}) - T(x) - T(y) = 0$$

for all $x,\ y \in X$ with $x \perp y$. Hence $T : X \to Y$ is an orthogonally Jensen additive mapping. If we put $y = 0$ in the last equation, then we have

$$2T\left(\frac{x+0}{2}\right) - T(x) - T(0) = 0$$

$$2T(\frac{x}{2}) - T(x) = 0$$

$$2T(\frac{x}{2}) = T(x)$$

and by replacing x for $2x$ in the last equation and dividing the resulting equation by 2, we have

$$T(x) = \frac{1}{2}T(2x). \qquad (a)$$

Now we show by induction that

$$\frac{1}{2^n}T(2^n x) = T(x). \qquad (b)$$

Since $\frac{1}{2^1}T(2^1 x) = \frac{1}{2}T(2x) = T(x) \quad by\ (a)$.
Hence (b) is true for $n = 1$.
Let (b) is true for $n = k$, we get

$$\frac{1}{2^k}T(2^k x) = T(x). \qquad (c)$$

We show that (b) is true for $n = k+1$.

$$\begin{aligned} \frac{1}{2^{k+1}}T(2^{k+1}x) &= \frac{1}{2^k.2}T(2^k.2x) \quad let\ y = 2x \\ &= \frac{1}{2}[\frac{1}{2^k}T(2^k y)] \\ &= \frac{1}{2}T(y)\ by\ (c) \\ &= \frac{1}{2}T(2x) = T(x)\ by\ (a). \end{aligned}$$

Hence $\frac{1}{2^n}T(2^n x) = T(x)$ for all $x \in X$ and $n \in \mathbb{N}$.

To prove the uniqueness, let $L : X \to Y$ to be another orthogonally Jensen additive such that $\|f(x) - L(x)\|_Y \leq \frac{2^p\theta}{2-2^p}\|x\|_X^p$. Then

$$
\begin{aligned}
\|T(x) - L(x)\|_Y = & \frac{1}{2^n}\|T(2^n x) - L(2^n x)\|_Y \quad \textit{by (b)} \\
= & \frac{1}{2^n}\|T(2^n x) - f(2^n x) + f(2^n x) - L(2^n x)\|_Y \\
\leq & \frac{1}{2^n}(\|f(2^n x) - T(2^n x)\|_Y + \|f(2^n x) - L(2^n x)\|_Y) \\
\leq & \frac{1}{2^n}(\frac{2^p\theta}{2-2^p}\|2^n x\|_X^p + \frac{2^p\theta}{2-2^p}\|2^n x\|_X^p) \\
= & \frac{2}{2^n}(\frac{2^p\theta}{2-2^p}\|2^n x\|_X^p) \\
= & \frac{2^{pn}}{2^n}(\frac{2.2^p\theta}{2-2^p}\|x\|_X^p) = \frac{2^{p+1}\theta}{2-2^p}.\frac{2^{pn}}{2^n}\|x\|_X^p \\
\leq & \frac{2^{p+1}\theta}{2-2^p}\|x\|_X^p \lim_{n\to\infty} \frac{2^{pn}}{2^n} \\
= & \frac{2^{p+1}\theta}{2-2^p}\|x\|_X^p \lim_{n\to\infty} 2^{(p-1)n} = 0
\end{aligned}
$$

since $|2^{p-1}| < 1$ for $p < 1$ and $\lim_{n\to\infty} 2^{(p-1)n} = 0$ for all $x \in X$ and $n \in \mathbb{N}$. We conclude that $T(x) = L(x)$, for all $x \in X$, which proves the uniqueness of T. □

Theorem 2.1.4. *[1] Let θ and p $(p > 1)$ be nonnegative real numbers. Suppose that $f : X \to Y$ is a mapping with $f(0) = 0$ fulfilling*

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\|_Y \leq \theta(\|x\|_X^p + \|y\|_X^p) \tag{2.5}$$

for all $x, y \in \mathbb{X}$ with $x \perp y$. Then there exists a unique orthogonally Jensen additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\|_Y \leq \frac{2^p\theta}{2^p - 2}\|x\|_X^p \tag{2.6}$$

for all $x \in X$.

Proof. Similarly as above we have from (2.3) the following

$$\|2^n f(\frac{x}{2^n}) - 2^m f(\frac{x}{2^m})\|_Y \leq \theta \sum_{k=n}^{m-1} \frac{2^k}{2^{pk}}\|x\|_X^p \tag{2.7}$$

for all nonnegative integers n, m with $n < m$.

Let us define a mapping $T : X \to Y$ by

$$T(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. The function T is well defined because Y is a Banach space and the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in Y. Letting $n = 0$ and $m \to \infty$ in (2.7), we get the inequality (2.6) as the following

$$\begin{aligned}
\lim_{m\to\infty} \|2^0 f(\frac{x}{2^0}) - 2^m f(\frac{x}{2^m})\|_Y &= \|f(x) - T(x)\| \\
&\leq \lim_{m\to\infty} \theta \sum_{k=0}^{m-1} \frac{2^k}{2^{pk}} \|x\|_X^p \\
&= \theta \|x\|_X^p \lim_{m\to\infty} \sum_{k=0}^{m-1} 2^{(1-p)k} \\
&= \theta \|x\|_X^p \ \frac{1}{1-\frac{2}{2^p}} \ = \ \theta \|x\|_X^p \ \frac{1}{\frac{2^p-2}{2^p}} \ \textit{since } p > 1 \\
&= \theta \|x\|_X^p \ \frac{2^p}{2^p - 2}
\end{aligned}$$

Then for any $n \in \mathbb{N}$, and from homogeneity of orthogonal relation one has $\frac{1}{2^n}\ x \perp \frac{1}{2^n}\ y$, whence

$$\begin{aligned}
\|2T(\frac{x+y}{2}) - T(x) - T(y)\|_Y &= \lim_{n\to\infty} \|2(2^n f(\frac{x+y}{2^n})) - 2^n f(\frac{x}{2^n}) - 2^n f(\frac{y}{2^n})\|_Y \\
&= \lim_{n\to\infty} 2^n \|2f(\frac{x+y}{2^n}) - f(\frac{x}{2^n}) - f(\frac{y}{2^n})\|_Y \\
&\leq \lim_{n\to\infty} 2^n \theta (\|\frac{x}{2^n}\|_X^p + \|\frac{y}{2^n}\|_X^p) \\
&= \lim_{n\to\infty} \frac{2^n \theta}{2^{np}} (\|x\|_X^p + \|y\|_X^p) \\
&= \lim_{n\to\infty} 2^{(1-p)n} \theta (\|x\|_X^p + \|y\|_X^p) \\
&\leq \theta (\|x\|_X^p + \|y\|_X^p) \lim_{n\to\infty} 2^{(1-p)n} \ = \ 0
\end{aligned}$$

for all $x,\ y \in X$ with $x \perp y$, where for $p > 1$, $|1-p| < 1$ and $\lim_{n\to\infty} 2^{(1-p)n} = 0$. So

$$2T(\frac{x+y}{2}) - T(x) - T(y) = 0$$

for all $x, y \in X$ with $x \perp y$. Hence $T : X \to Y$ is orthogonally Jensen additive mapping. Also, as above we can show the following

$$2^n T(\frac{x}{2^n}) = T(x)$$

for all $x \in X,\ n \in \mathbb{N}$.

To prove the uniqueness, let $L : X \to Y$ to be another orthogonally Jensen additive such that $\|f(x) - L(x)\|_Y \le \frac{2^p\theta}{2^p-2}\|x\|_X^p$. Then

$$\begin{aligned}
\|T(x) - L(x)\|_Y &= 2^n\|T(\frac{x}{2^n}) - L(\frac{x}{2^n})\|_Y \\
&= 2^n\|T(\frac{x}{2^n}) - f(\frac{x}{2^n}) + f(\frac{x}{2^n}) - L(\frac{x}{2^n})\|_Y \\
&\le 2^n\|f(\frac{x}{2^n}) - T(\frac{x}{2^n})\| + \|f(\frac{x}{2^n}) - L(\frac{x}{2^n})\|_Y \\
&\le 2^n\left(\frac{2^p\theta}{2^p-2}\frac{\|x\|_X^p}{2^{pn}} + \frac{2^p\theta}{2^p-2}\frac{\|x\|_X^p}{2^{pn}}\right) \\
&= 2^n(\frac{2.2^p\theta}{2^p-2})\frac{\|x\|_X^p}{2^{pn}} \\
&= \frac{2^{p+1}\theta}{2^p-2}.\frac{2^n}{2^{pn}}\|x\|_X^p \\
&\le \frac{2^{p+1}\theta}{2^p-2}\|x\|_X^p \lim_{n\to\infty} 2^{(1-p)n} = 0
\end{aligned}$$

since for $p > 1$, $|1-p| < 1$ and $\lim_{n\to\infty} 2^{(1-p)n} = 0$. So we have $T(x) = L(x)$ for all $x \in X$. This proves the uniqueness of T. □

2.2 Stability of the orthogonally Jensen quadratic functional equation

In this section we prove the generalized Hyers-Ulam stability of the orthogonally Jensen additive functional equation

$$2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) = f(x) + f(y)$$

for all $x, y \in X$ with $x \perp y$.

Throughout this section, $(X, \perp)$ denotes an orthogonality normed space with norm $\| \,.\, \|_X$ and $(Y, \| \,.\, \|_Y)$ is a Banach space.

Theorem 2.2.1. *[1] Let θ and p $\quad(p < 2)$ be nonnegative real numbers. Suppose that $f : X \to Y$ is a mapping with $f(0) = 0$ fulfilling*

$$\|2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) - f(x) - f(y)\|_Y \le \theta(\|x\|_X^p + \|y\|_X^p) \tag{2.8}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally Jensen quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \le \frac{2^p\theta}{4-2^p}\|x\|_X^p \tag{2.9}$$

for all $x \in X$.

Proof. : Putting $y = 0$ in (2.8), we get

$$\|2f(\frac{x+0}{2}) + 2f(\frac{x-0}{2}) - f(x) - f(0)\|_Y \leq \theta(\|x\|_X^p + \|0\|_X^p)$$

since $f(0) = 0$, then

$$\|4f(\frac{x}{2}) - f(x)\|_Y \leq \theta\|x\|_X^p \tag{2.10}$$

for all $x \in X$, with $x \perp 0$. So

$$\|f(\frac{x}{2}) - \frac{1}{4}f(x)\|_Y \leq \frac{\theta}{4}\|x\|_X^p$$

replacing x by $2x$ in above inequality, we have

$$\begin{aligned}\|f(x) - \frac{1}{4}f(2x)\|_Y &\leq \frac{\theta}{4}\|2x\|_X^p \\ &= \frac{2^p\theta}{4}\|x\|_X^p\end{aligned}$$

for all $x \in X$. And similarly as above we have

$$\|\frac{1}{4^n}f(2^n x) - \frac{1}{4^m}f(2^m x)\|_Y \leq \frac{2^p\theta}{4}\sum_{k=n}^{m-1}\frac{2^{pk}}{4^k}\|x\|_X^p \tag{2.11}$$

for all nonnegative integers n, m with $n < m$. Thus $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence in Y. Since Y is complete, there exists a mapping $Q : X \to Y$ defined by

$$Q(x) = \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Letting $n = 0$ and $m \to \infty$ in (2.11), we get the inequality (2.9) as following

$$\begin{aligned}\lim_{m\to\infty}\|\frac{1}{4^0}f(2^0 x) - \frac{1}{4^m}f(2^m x)\|_Y &= \|f(x) - Q(x)\|_Y \\ &\leq \lim_{m\to\infty}\frac{2^p\theta}{4}\sum_{k=0}^{m-1}\frac{2^{pk}}{4^k}\|x\|_X^p \\ &= \frac{2^p\theta}{4}\|x\|_X^p\lim_{m\to\infty}\sum_{k=0}^{m-1}\frac{2^{pk}}{4^k} \\ &= \frac{2^p\theta}{4}\|x\|_X^p\frac{1}{1-\frac{2^p}{4}} \\ &= \frac{2^p\theta}{4}\|x\|_X^p\frac{4}{4-2^p} \\ &= \frac{2^p\theta}{4-2^p}\|x\|_X^p\end{aligned}$$

To show that Q is orthogonally Jensen quadratic, choose arbitrarily $x, y \in X$, $x \perp y$, then for any $n \in \mathbb{N}$, and from homogeneity of orthogonal relation one has $2^n x \perp 2^n y$, and by (2.8) we have

$$\begin{aligned}
&\|2Q\ (\frac{x+y}{2}) + 2Q\ (\frac{x-y}{2}) - Q(x) - Q(y)\|_Y \\
= &\ \| \lim_{n\to\infty} \frac{1}{4^n} 2f(\frac{2^n x + 2^n y}{2}) + \lim_{n\to\infty} \frac{1}{4^n} 2f(\frac{2^n x - 2^n y}{2}) - \lim_{n\to\infty} \frac{1}{4^n} f(2^n x) - \lim_{n\to\infty} \frac{1}{4^n} f(2^n y)\|_Y \\
= &\ \lim_{n\to\infty} \frac{1}{4^n} \|2f(\frac{2^n x + 2^n y}{2}) + 2f(\frac{2^n x - 2^n y}{2}) - f(2^n x) - f(2^n y)\|_Y \\
\leq &\ \lim_{n\to\infty} \frac{\theta}{4^n} (\|2^n x\|_X^p + \|2^n y\|_X^p) \\
= &\ \theta(\|x\|_X^p + \|y\|_X^p) \lim_{n\to\infty} \frac{2^{pn}}{4^n} \\
= &\ \theta(\|x\|_X^p + \|y\|_X^p) \lim_{n\to\infty} 2^{(p-2)n} = 0
\end{aligned}$$

where for $p < 2$, $|2^{(p-2)}| < 1$ and $\lim_{n\to\infty} 2^{(p-2)n} = 0$. for all $x, y \in X$ with $x \perp y$. So

$$2Q(\frac{x+y}{2}) + 2Q(\frac{x-y}{2}) - Q(x) - Q(y) = 0$$

for all $x, y \in X$ with $x \perp y$. Hence $Q : X \to Y$ is an orthogonally Jensen quadratic mapping. In the same way as previously, we can prove the following

$$\frac{1}{4^n} Q(2^n x) = Q(x),$$

for all $x \in X,\ n \in \mathbb{N}$.

To prove the uniqueness, Let $L : X \to Y$ be another orthogonally Jensen quadratic mapping satisfying (2.9). Then

$$\begin{aligned}
\|Q(x) - L(x)\|_Y &= \frac{1}{4^n} \|Q(2^n x) - L(2^n x)\|_Y \\
&= \frac{1}{4^n} (\|Q(2^n x) - f(2^n x) + f(2^n x) - L(2^n x)\|_Y) \\
&\leq \frac{1}{4^n} (\|f(2^n x) - Q(2^n x)\|_Y + \|f(2^n x) - L(2^n x)\|_Y) \\
&= \frac{2}{4^n} (\frac{2^p \theta}{4 - 2^p} \|2^n x\|_X^p) \\
&= \frac{2}{4^n} (\frac{2^p \theta}{4 - 2^p} \|2^n x\|_X^p) \\
&= \frac{2^{p+1}\theta}{4 - 2^p} . \frac{2^{pn}}{4^n} \|x\|_X^p \\
&\leq \frac{2^{p+1}\theta}{4 - 2^p} \|x\|_X^p \lim_{n\to\infty} 2^{(p-2)n} = 0
\end{aligned}$$

where for $p < 2$, $|2^{(p-2)n}| < 1$, and $\lim_{n\to\infty} 2^{(p-2)n} = 0$ for all $x \in X$. So we have $Q(X)$=$L(X)$ for all $x \in X$.
This proves the uniqueness of Q. $\square$

Theorem 2.2.2. *[1] Let θ and p $(p > 2)$ be nonnegative real numbers. Suppose that $f : X \to Y$ is a mapping with $f(0) = 0$ fulfilling*

$$\|2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) - f(x) - f(y)\|_Y \leq \theta(\|x\|_X^p + \|y\|_X^p) \tag{2.12}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally Jensen quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \leq \frac{2^p\theta}{2^p - 4}\|x\|_X^p \tag{2.13}$$

for all $x \in X$.

Proof. And similarly as above, it follows from (2.10) that

$$\|4^n f(\frac{x}{2^n}) - 4^m f(\frac{x}{2^m})\|_Y \leq \theta \sum_{k=n}^{m-1} \frac{4^k}{2^{pk}}\|x\|_X^p \tag{2.14}$$

for all nonnegative integers n, m with $n < m$. Thus $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in Y. Since Y is complete, there exists a mapping $Q : X \to Y$ defined by

$$Q(x) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Letting $n = 0$ and $m \to \infty$ in (2.14), we get the inequality (2.13) as following

$$\begin{aligned}
\lim_{m\to\infty} \|4^0 f(\frac{x}{2^0}) - 4^m f(\frac{x}{2^m})\|_Y &= \|f(x) - Q(x)\|_Y \\
&\leq \lim_{m\to\infty} \theta \sum_{k=0}^{m-1} \frac{4^k}{2^{pk}}\|x\|_X^p \\
&= \theta\|x\|_X^p \lim_{m\to\infty} \sum_{k=0}^{m-1} \frac{4^k}{2^{pk}} \\
&= \theta\|x\|_X^p \frac{1}{1 - \frac{4}{2^p}} \\
&= \theta\|x\|_X^p \frac{2^p}{2^p - 4} \\
&= \frac{2^p\theta}{2^p - 4}\|x\|_X^p
\end{aligned}$$

In order to show that Q is orthogonally Jensen quadratic, choose arbitrarily $x, y \in X$, $x \perp y$. Then for any $n \in \mathbb{N}$, and from homogeneity of orthogonal relation one has $\frac{1}{2^n}x \perp \frac{1}{2^n}y$, It follows from (2.12) that

$$\begin{aligned}
&\|2Q\,(\frac{x+y}{2}) + 2Q\,(\frac{x-y}{2}) - Q(x) - Q(y)\|_Y \\
&= \|2\lim_{n\to\infty} 4^n f\left(\frac{\frac{x}{2^n} + \frac{y}{2^n}}{2}\right) + 2\lim_{n\to\infty} 4^n f\left(\frac{\frac{x}{2^n} - \frac{y}{2^n}}{2}\right) - \lim_{n\to\infty} 4^n f(\frac{x}{2^n}) - \lim_{n\to\infty} 4^n f(\frac{y}{2^n})\|_Y
\end{aligned}$$

$$= \lim_{n\to\infty} 4^n \| 2f\left(\frac{\frac{x}{2^n}+\frac{y}{2^n}}{2}\right) + 2f\left(\frac{\frac{x}{2^n}-\frac{y}{2^n}}{2}\right) - f(\frac{x}{2^n}) - f(\frac{y}{2^n})\|_Y$$

$$\leq \lim_{n\to\infty} \theta 4^n (\|\frac{x}{2^n}\|_X^p + \|\frac{y}{2^n}\|_X^p)$$

$$= \theta(\|x\|_X^p + \|y\|_X^p) \lim_{n\to\infty} \frac{4^n}{2^{pn}}$$

$$= \theta(\|x\|_X^p + \|y\|_X^p) \lim_{n\to\infty} 2^{(2-p)n} = 0$$

where for $p > 2$, $|2^{(2-p)}| < 1$ and $\lim_{n\to\infty} 2^{(2-p)n} = 0$. for all $x, y \in X$ with $x \perp y$. So

$$2Q(\frac{x+y}{2}) + 2Q(\frac{x-y}{2}) - Q(x) - Q(y) = 0$$

for all $x, y \in X$ with $x \perp y$. Hence $Q : X \to Y$ is an orthogonally Jensen quadratic mapping. In the same way as previously, we can prove the following

$$4^n Q(\frac{x}{2^n}) = Q(x),$$

for all $x \in X,\ n \in \mathbb{N}$.

To prove the uniqueness, Let $L : X \to Y$ be another orthogonally Jensen quadratic mapping satisfying (2.13) Then

$$\begin{aligned}
\|Q(x) - L(x)\|_Y = 4^n \|Q(\frac{x}{2^n}) - L(\frac{x}{2^n})\|_Y& \\
&\leq 4^n(\|f(\frac{x}{2^n}) - Q(\frac{x}{2^n})\|_Y + \|f(\frac{x}{2^n}) - L(\frac{x}{2^n})\|_Y) \\
&= 4^n(\frac{2^p\theta}{2^p-4}\|\frac{x}{2^n}\|_X^p) + (\frac{2^p\theta}{2^p-4}\|\frac{x}{2^n}\|_X^p) \\
&= 2.4^n(\frac{2^p\theta}{2^p-4}\|\frac{x}{2^n}\|_X^p) \\
&= \frac{2^{p+1}\theta}{2^p-4} . \frac{4^n}{2^{pn}}\|x\|_X^p \\
&\leq \frac{2^{p+1}\theta}{2^p-4}\|x\|_X^p \lim_{n\to\infty} 2^{(2-p)n} = 0
\end{aligned}$$

since ($p > 2$), $|2^{(2-p)n}| < 1$ for all $x \in X$ and $n \in \mathbb{N}$. So we have $Q(X)$=$L(X)$ for all $x \in X$.

This proves the uniqueness of Q. □

Chapter 3

The Stability of the Pexiderized Cosine Functional Equation

This chapter consists of two sections. In the first one we will present the concept of the superstability and some of the examples described him.
In the second section we discuss the superstability of the pexiderized cosine functional equation

$$f_1(x+y) + f_2(x-y) = 2g_1(x)g_2(y),$$

where f_1, f_2, g_1, and g_2 are functions from $\mathbb{R}$ to $\mathbb{C}$. Where Kusollerschariya and Nakmahachalasint get some new results will be explained here in [3].

3.1 Introduction

There are cases in which each approximate homomorphism is actually a true homomorphism. In such cases, we call the equation of homomorphism **superstable**. J.A. Baker, J. Lawrence and F. Zoritto (1979) introduced by [3], the following concept for the superstable

Definition 3.1.1. [3] Let f be a function from a Banach space to a Banach space which satisfies the inequality $|E_1(f) - E_2(f)| \leq \epsilon$, then either f is bounded or $E_1(f) = E_2(f)$, where $E_1(f), E_2(f)$ is the left, right hand said respectively of the given functional equation.

This concept is now known as the superstability.

Example 3.1.2. *[13] If a real-valued function f defined on a real vector space satisfies the functional inequality $|f(x+y) - f(x)f(y)| \leq \delta$, for some $\delta > 0$ and for all x and y, then f is either bounded or $f(x+y) = f(x)f(y)$.*

Example 3.1.3. *[13] Let* $(G, .)$ *be a semigroup and let* $\delta > 0$ *be given. If a function* $f : G \to \mathbb{C}$ *satisfies the inequality* $|f(x.y) - f(x)f(y)| \le \delta$, *for all* $x, y \in G$, *then either* $|f(x)| \le (1 + \sqrt{1 + 4\delta})/2$ *for all* $x \in G$ *or* $f(x.y) = f(x)f(y)$ *for all* $x,\ y \in G$.

Example 3.1.4. *[13] Let* $(G, .)$ *be an abelian group and let* $\delta > 0$ *If a function* $f : G \to \mathbb{C}$ *satisfies the inequality* $|f(x + y) + f(x - y) - 2f(x)f(y)| \le \delta$, *for all* $x, y \in G$, *and for some* $\delta > 0$ *then either* $|f(x)| \le (1 + \sqrt{1 + 2\delta})/2$ *for any* $x \in G$ *or* $f(x + y) + f(x - y) = 2f(x)f(y)$ *for all* $x,\ y \in G$.

In the previous cases we say that the equations is superstable. Hence we can consider the superstability is a special case of the Hyers-Ulam stability problem.

3.2 The superstability of the pexiderized cosine functional equation

In this section we investigate the superstability of the pexiderized cosine functional equation

$$f_1(x+y)+f_2(x-y)=2g_1(x)g_2(y),$$

where f_1, f_2, g_1, and g_2 are functions from $\mathbb{R}$ to $\mathbb{C}$.

Theorem 3.2.1. *[3] Let $f_1, f_2, g_1, g_2 : \mathbb{R} \to \mathbb{C}$ be functions satisfying*

$$|f_1(x+y)+f_2(x-y)-2g_1(x)g_2(y)| \leq \delta \tag{3.1}$$

for all x,$y \in \mathbb{R}$. Then either g_1 is bounded or there exists an even function $h : \mathbb{R} \to \mathbb{C}$ with $h(0) = 1$ such that

$$g_2(x+y)+g_2(x-y)=2g_2(x)h(y) \qquad \textit{for all } x,y \in \mathbb{R}.$$

Proof. : Suppose that g_1 is unbounded. Then we can choose a sequence $\{x_n\}$ such that $0 \neq |g_1(x_n)| \to \infty$ as $n \to \infty$. For each $n \in \mathbb{N}$, setting $x = x_n$ in (3.1) and dividing both sides of the resulting inequality by $|2g_1(x_n)|$, we have

$$\frac{|f_1(x_n+y)+f_2(x_n-y)-2g_1(x_n)g_2(y)|}{|2g_1(x_n)|} \leq \frac{\delta}{|2g_1(x_n)|}$$

then

$$\left|\frac{f_1(x_n+y)+f_2(x_n-y)}{2g_1(x_n)} - g_2(y)\right| \leq \frac{\delta}{|2g_1(x_n)|}$$

since $|g_1(x_n)| \to \infty$, then the right-hand side approaches to 0 as $n \to \infty$. Therefore,

$$g_2(y) = \lim_{n\to\infty} \frac{f_1(x_n+y)+f_2(x_n-y)}{2g_1(x_n)} \qquad \forall y \in \mathbb{R}. \tag{3.2}$$

Substituting x by $x_n + y$, y by x in (3.1). We obtained

$$|f_1((x_n+y)+x)+f_2((x_n+y)-x)-2g_1(x_n+y)g_2(x)| \leq \delta.$$

Also, Substituting x by $x_n - y$, y by x in (3.1). We obtained

$$|f_1((x_n-y)+x)+f_2((x_n-y)-x)-2g_1(x_n-y)g_2(x)| \leq \delta.$$

By the triangle inequality, the last two inequalities lead to

$$\begin{aligned}&|f_1((x_n+y)+x)+f_2((x_n+y)-x)-2g_1(x_n+y)g_2(x)\\ &\quad +f_1((x_n-y)+x)+f_2((x_n-y)-x)-2g_1(x_n-y)g_2(x)| \leq\\ &\quad\quad |f_1((x_n+y)+x)+f_2((x_n+y)-x)-2g_1(x_n+y)g_2(x)|+\end{aligned}$$

$$
\begin{aligned}
&|f_1((x_n - y) + x) + f_2((x_n - y) - x) - 2g_1(x_n - y)g_2(x)| \\
&\leq \delta + \delta \\
&= 2\delta.
\end{aligned}
$$

So,

$$
\begin{aligned}
&|f_1(x_n + (x + y)) + f_2(x_n - (x + y)) \\
&\quad + f_1(x_n + (x - y)) + f_2(x_n - (x - y)) \\
&\qquad - 2g_1(x_n + y)g_2(x) - 2g_1(x_n - y)g_2(x)| \leq 2\delta.
\end{aligned}
$$

By dividing both said of the last inequalities by $|2g_1(x_n)|$, we have

$$
\begin{aligned}
&\Bigg| \frac{f_1(x_n + (x + y)) + f_2(x_n - (x + y))}{2g_1(x_n)} \\
&\quad + \frac{f_1(x_n + (x - y)) + f_2(x_n - (x - y))}{2g_1(x_n)} \\
&\qquad - 2\left(\frac{g_1(x_n + y) + g_1(x_n - y)}{2g_1(x_n)}\right) g_2(x) \Bigg| \leq \frac{2\delta}{|2g_1(x_n)|}. \qquad (3.3)
\end{aligned}
$$

Since $0 \neq |g_1(x_n)| \to \infty$, then The right-hand side converges to 0 as $n \to \infty$. So we define

$$
h(y) = \lim_{n\to\infty} \frac{g_1(x_n + y) + g_1(x_n - y)}{2g_1(x_n)} \quad for\ all \quad y \in \mathbb{R}.
$$

Notice that h is even and $h(0) = 1$, we can see that by the following

$$
\begin{aligned}
h(-y) &= \lim_{n\to\infty} \frac{g_1(x_n + (-y)) + g_1(x_n - (-y))}{2g_1(x_n)} \\
&= \lim_{n\to\infty} \frac{g_1(x_n - y) + g_1(x_n + y)}{2g_1(x_n)} \\
&= h(y).
\end{aligned}
$$

Also,

$$
\begin{aligned}
h(0) &= \lim_{n\to\infty} \frac{g_1(x_n + 0) + g_1(x_n - 0)}{2g_1(x_n)} \\
&= \lim_{n\to\infty} \frac{g_1(x_n) + g_1(x_n)}{2g_1(x_n)} \\
&= \lim_{n\to\infty} \frac{2g_1(x_n)}{2g_1(x_n)} = 1.
\end{aligned}
$$

Then, by letting $n \to \infty$ in (3.3), we see that

$$
\begin{aligned}
\lim_{n\to\infty} &\Bigg| \frac{f_1(x_n + (x + y)) + f_2(x_n - (x + y))}{2g_1(x_n)} \\
&\quad + \frac{f_1(x_n + (x - y)) + f_2(x_n - (x - y))}{2g_1(x_n)}
\end{aligned}
$$

$$-2\left(\frac{g_1(x_n+y)+g_1(x_n-y)}{2g_1(x_n)}\right)g_2(x)\Bigg| \le \lim_{n\to\infty}\frac{2\delta}{|2g_1(x_n)|}$$

then,

$$\Bigg|\lim_{n\to\infty}\frac{f_1(x_n+(x+y))+f_2(x_n-(x+y))}{2g_1(x_n)}$$
$$+\lim_{n\to\infty}\frac{f_1(x_n+(x-y))+f_2(x_n-(x-y))}{2g_1(x_n)}$$
$$-2\lim_{n\to\infty}\left(\frac{g_1(x_n+y)+g_1(x_n-y)}{2g_1(x_n)}\right)g_2(x)\Bigg| \le \lim_{n\to\infty}\frac{2\delta}{|2g_1(x_n)|}.$$

Then, from (3.2) and by definition of a function h above we have

$$g_2(x+y)+g_2(x-y)-2g_2(x)h(y)=0 \qquad \text{for all } x,y\in\mathbb{R}.$$

So,

$$g_2(x+y)+g_2(x-y)=2g_2(x)h(y) \quad \text{for all } x,y\in\mathbb{R}$$

as desired.

In the other way around, we look at the case g_2 is bounded. $\square$

Theorem 3.2.2. *[3] Let $f_1, f_2, g_1, g_2 : \mathbb{R}\to\mathbb{C}$ be functions satisfying*

$$|f_1(x+y)+f_2(x-y)-2g_1(x)g_2(y)|\le\delta, \tag{3.4}$$

for all $x, y\in\mathbb{R}$. Then either g_2 is bounded or there exists an even function $h:\mathbb{R}\to\mathbb{C}$ with $h(0)=1$ such that

$$g_1(x+y)+g_1(x-y)=2g_1(x)h(y) \qquad for\ all \quad x,\ y\in\mathbb{R}.$$

Proof. :Suppose that g_2 is unbounded. Then we can choose a sequence $\{y_n\}$ such that $0\neq|g_2(y_n)|\to\infty$ as $n\to\infty$. It can be shown similarly to the above theorem that

$$g_1(x)=\lim_{n\to\infty}\frac{f_1(y_n+x)+f_2(y_n-x)}{2g_2(y_n)} \quad for\ all\ x\in\mathbb{R}. \tag{3.5}$$

We set $(x,y)=(y_n+y,x)$ and $(x,y)=(y_n-y,x)$, respectively, in (3.4) and proceed the same fashion of the previous proof. We are led to defining a function h as follows

$$h(y)=\lim_{n\to\infty}\frac{g_2(y_n+y)+g_2(y_n-y)}{2g_2(y_n)} \quad \forall y\in\mathbb{R}.$$

and we then have

$$g_1(x+y)+g_1(x-y)=2g_1(x)h(y) \quad \text{for all } x,y\in\mathbb{R}$$

Also, note that h is even and $h(0)=1$. $\square$

Remark 3.2.3. [3] Notice that if we further assume the evenness of f , then either $f \equiv 0$ or $\hat{f} = \frac{f(x)}{f(0)}$ is equal to the cosine, the cosine hyperbolic, or the constant function 1 which satisfies the cosine functional equation.

Lemma 3.2.4. *[3] $f, g : \mathbb{R} \to \mathbb{C}$ be functions satisfying*

$$f(x+y) + f(x-y) = 2f(x)g(y) \quad for\ all \quad x, y \in \mathbb{R}.$$

If f is an even function, then either $f \equiv 0$ or $\hat{f}(x)$ satisfies (DE).

Proof. Put $x = 0$ in above equation we have

$$f(0+y) + f(0-y) = 2f(0)g(y) \;\; for\ all \quad y \in \mathbb{R},$$

since f is even function, then

$$2f(y) = 2f(0)g(y),$$

also,

$$g(y) = \frac{f(y)}{f(0)}.$$

Suppose $f \not\equiv 0$. Then we choose $\hat{f}(x) = \frac{f(x)}{f(0)}$ since $f(0) \neq 0$.

$$\frac{f(x+y)}{f(0)} + \frac{f(x-y)}{f(0)} = \frac{2f(x)f(y)}{f(0)f(0)}$$

then

$$\hat{f}(x+y) + \hat{f}(x-y) = 2\hat{f}(x)\hat{f}(y).$$

Hence $\hat{f}$ satisfies (DE). □

Theorem 3.2.5. *[3] Let $f_1, f_2, g_1, g_2 : \mathbb{R} \to \mathbb{C}$ be functions such that*

$$|f_1(x+y) + f_2(x-y) - 2g_1(x)g_2(y)| \leq \delta,$$

for all $x, y \in \mathbb{R}$. Suppose that g_2 is an even function and $g_2 \not\equiv 0$. Then either g_1 is bounded or $\hat{g}_2$ satisfies (DE).

Proof. : Assume that g_1 is unbounded. It follows from Theorem (3.2.1) that there is a function h such that, for every $x, y \in \mathbb{R}$

$$g_2(x+y) + g_2(x-y) = 2g_2(x)h(y)$$

Since g_2 is even function and $g_2 \not\equiv 0$. And by above lemma we have either $g_2 \equiv 0$ or $\hat{g}_2$ satisfies (DE) but $g_2 \not\equiv 0$, then $\hat{g}_2$ satisfies (DE). □

Theorem 3.2.6. *[3] Let $f_1, f_2, g_1, g_2 : \mathbb{R} \to \mathbb{C}$ be functions such that*

$$|f_1(x+y)+f_2(x-y)-2g_1(x)g_2(y)| \leq \delta,$$

for all $x, y \in \mathbb{R}$. Suppose that g_1 is an even function and $g_1 \not\equiv 0$. Then either g_2 is bounded or $\hat{g}_1$ satisfies (DE).

Proof. Suppose that g_2 is unbounded. It follows from Theorem (3.2.2) that there is a function h such that, for every $x, y \in \mathbb{R}$

$$g_1(x+y)+g_1(x-y)=2g_1(x)h(y)$$

Since g_1 is even function and $g_1 \not\equiv 0$. Also, by lemma 1 we have either $g_1 \equiv 0$ or $\hat{g}_1$ satisfies (DE) but $g_1 \not\equiv 0$, then $\hat{g}_1$ satisfies (DE). □

Corollary 3.2.7. *[3] Let $f, g_1, g_2 : \mathbb{R} \to \mathbb{C}$ be functions such that*

$$|f(x+y)+f(x-y)-2g_1(x)g_2(y)| \leq \delta,$$

for all $x, y \in \mathbb{R}$. Suppose that $g_2 \not\equiv 0$. Then either g_1 is bounded or $\hat{g}_2$ satisfies (DE).

Proof. :Taking $f_1 = f_2 = f$ in Theorem (3.2.5), we infer the evenness of g_2 by replacing (y) by $(-y)$ from its definition in (3.2) we get

$$\begin{aligned}
g_2(-y) &= \lim_{n\to\infty} \frac{f_1(x_n+(-y))+f_2(x_n-(-y))}{2g_1(x_n)} \\
&= \lim_{n\to\infty} \frac{f_1(x_n-y)+f_2(x_n+y)}{2g_1(x_n)} \\
&= \lim_{n\to\infty} \frac{f(x_n-y)+f(x_n+y)}{2g_1(x_n)} \\
&= \lim_{n\to\infty} \frac{f(x_n+y)+f(x_n-y)}{2g_1(x_n)} \\
&= \lim_{n\to\infty} \frac{f_1(x_n+y)+f_2(x_n-y)}{2g_1(x_n)} \\
&= g_2(y).
\end{aligned}$$

which completes the proof. □

Corollary 3.2.8. *[3] Let $f, g_1, g_2 : \mathbb{R} \to \mathbb{C}$ be functions such that*

$$|f(x+y)+f(x-y)-2g_1(x)g_2(y)| \leq \delta,$$

for all $x, y \in \mathbb{R}$. Suppose that f is even and $g_1 \not\equiv 0$. Then either g_2 is bounded or $\hat{g}_1$ satisfies (DE).

Proof. Taking $f_1 = f_2 = f$ in Theorem (3.2.5), The evenness of f and the definition of g_1 in (3.5) lead to the evenness of g_1 as following

$$\begin{aligned} g_1(-x) &= \lim_{n\to\infty} \frac{f_1((-x)+y_n)+f_2((-x)-y_n)}{2g_n(y_n)} \\ &= \lim_{n\to\infty} \frac{f_1(-(x-y_n))+f_2(-(x+y_n))}{2g_n(y_n)} \\ &= \lim_{n\to\infty} \frac{f(-(x-y_n))+f(-(x+y_n))}{2g_n(y_n)} \\ &= \lim_{n\to\infty} \frac{f(x+y_n)+f(x-y_n)}{2g_n(y_n)} \\ &= \lim_{n\to\infty} \frac{f_1(x+y_n)+f_2(x-y_n)}{2g_n(y_n)} \\ &= g_1(x). \end{aligned}$$

which completes the proof. □

Corollary 3.2.9. *[3] Let $f, g : \mathbb{R} \to \mathbb{C}$ be functions such that*

$$|f(x+y)+f(x-y)-2f(x)g(y)| \le \delta,$$

for all $x, y \in \mathbb{R}$. Then either f is bounded or g satisfies (DE).

Proof. : Take $g_1 = f$ and $g_2 = g$ in definition of g_2 we have the evenness o as following

$$\begin{aligned} g(-y) &= \lim_{n\to\infty} \frac{f(x_n+(-y))+f(x_n-(-y))}{2f(x_n)} \\ &= \lim_{n\to\infty} \frac{f(x_n+y)+f(x_n-y)}{2f(x_n)} \\ &= g(y). \end{aligned}$$

Also,

$$\begin{aligned} g(0) &= \lim_{n\to\infty} \frac{f(x_n+0)+f(x_n-0)}{2f(x_n)} \\ &= \lim_{n\to\infty} \frac{f(x_n)+f(x_n)}{2f(x_n)} \\ &= \lim_{n\to\infty} \frac{2f(x_n)}{2f(x_n)} \\ &= 1. \end{aligned}$$

By remarks(3.2.3), since g is even and $g(0) = 1$, then $g \not\equiv 0$. Since $\hat{g} = \frac{g(x)}{g(0)} = \frac{g(x)}{1} = g$, by lemma (1) we have $\hat{g}$ satisfies $(D.E)$. Therefore g is equal to $\hat{g}$ satisfies (DE).
Also Take $g_1 = f$, $g_2 = g$ and in Corollary (3.2.7) we obtained
Let $f, g : \mathbb{R} \to \mathbb{C}$ be functions such that

$$|f(x+y)+f(x-y)-2f(x)g(y)| \le \delta,$$

for all $x, y \in \mathbb{R}$. Suppose that $g \not\equiv 0$. Then either f is bounded or $\hat{g}$ satisfies (DE), replacing $\hat{g}$ by g in above we complete the proof. $\square$

Corollary 3.2.10. *[3] Let $f, g : \mathbb{R} \to \mathbb{C}$ be functions such that*

$$|f(x+y) + f(x-y) - 2g(x)f(y)| \leq \delta,$$

for all $x, y \in \mathbb{R}$. Then provided that f is even, either f is bounded or g satisfies (DE).

Proof. : Take $f_1 = f_2 = f$, $g_1 = g$ in definition of g_1 we obtained the evenness of g by using the evenness of f, we have

$$\begin{aligned} g(-x) &= \lim_{n\to\infty} \frac{f((-x)+y_n) + f((-x)-y_n)}{2f(y_n)} \\ &= \lim_{n\to\infty} \frac{f(-(x-y_n)) + f(-(x+y_n))}{2f(y_n)} \\ &= \lim_{n\to\infty} \frac{f(x+y_n) + f(x-y_n)}{2f(y_n)} \\ &= g(x). \end{aligned}$$

Also,

$$\begin{aligned} g(0) &= \lim_{n\to\infty} \frac{f(0+y_n) + f(0-y_n)}{2f(y_n)} \\ &= \lim_{n\to\infty} \frac{f(+y_n) + f(-y_n)}{2f(y_n)} \\ &= \lim_{n\to\infty} \frac{2f(y_n)}{2f(y_n)} \\ &= 1. \end{aligned}$$

Thus by lemma (1), g is equal to $\hat{g}$, and satisfies (DE). Now take $g_1 = g$ and $g_2 = f$ in Corollary (3.2.8), we complete the proof. $\square$

Corollary 3.2.11. *[3] Let $f, g : \mathbb{R} \to \mathbb{C}$ be functions such that*

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \delta,$$

for all $x, y \in \mathbb{R}$. Then either f is bounded or f satisfies (DE).

Proof. Put $g = f$ in corollary (3.2.9) we have done. $\square$

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